Solving linear equations—row reduction

One method to put a matrix A into reduced form:

NOTE: The *leading* entry of a row of a matrix is the left-most nonzero entry of that row.

1. Be clever. Then, put all rows consisting entirely of zeros at the bottom of the matrix.
2. Choose a simple row with a leading entry in the first column. Switch this row with the first row. Use this row and elementary row operations to get all entries in the first column below the first row equal to zero. ("*clean down*" the first column).
3. Now, forget about this first row. Choose a simple row with a leading entry in the second column. Switch this row with the second row. Use this row and elementary row operations to make all entries below its leading entry equal to zero (clean down the second column).
4. Keep cleaning down successive columns.

   The matrix is now in triangular form (but not necessarily reduced). The leading entries are called the corner (or pivot) entries for A.
5. "*Clean up*". That is, use the lowest nonzero row and elementary row operations to make all entries in the column above its leading entry zero. Make that leading entry equal to one by scaling the row.
6. Now, forget about this lowest row and use the row above it to clean up, making all entries in the column above its leading entry zero. Then, make its leading entry equal to one by scaling the row.
7. Continue cleaning up until the matrix is reduced.

To solve $m \times n$ linear algebraic equations $A\boldsymbol{x} = \boldsymbol{b}$:

1. Form the augmented matrix $[A\,\boldsymbol{b}]$ and reduce.

2a. If in this process you get a corner entry in the last column (the matrix has a row: $(0, \ldots, 0, b)$ where $b \neq 0$) then the system is inconsistent. (That is, there are no solutions to the equations because one of the equations is $0 = b$.)

2b. If not, continue to reduce, convert to equations and solve for the *corner variables* (variables corresponding to corner entries of the reduced matrix) in terms of the *free variables* (noncorner variables) and constants. The noncorner variables are parameters and can be labeled a, b, c, and so on.

To solve the $m \times n$ homogeneous linear algebraic equations $A\boldsymbol{x} = \boldsymbol{0}$:

1. Reduce the augmented matrix $[A\,\boldsymbol{0}]$. (You can reduce A as long as you convert back to **homogeneous** equations.)
2. Solve for the corner variables in terms of free (noncorner) variables (if there are free variables).
3. If there are free variables: successively set each free variable equal to one (or an 'easy' nonzero number) and set the other free variables equal to zero. Then, write your solutions as vectors. This gives a list of **linearly independent** solution vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$(one for each free variable). Every solution to $A\boldsymbol{x} = \boldsymbol{0}$ is a linear combination of these solution vectors.