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Sign Diagrams

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In these notes we describe a method for deciding when a given polynomial or quotient of polynomials is positive, and when it is negative. The method assumes that we can factor the polynomials. It relies on the fact that the product of an odd number of negative factors is negative, while the product of an even number of negative factors is positive.

Let's begin by considering

$$r(x) = x^2 - 2x - 3 = (x+1)(x-3).$$

The factor $x-3$ is zero when x is 3, negative for x less than 3, and positive for x greater than 3. Figure 1 includes a flag going left from 3 showing where $x-3$ is negative. The factor $x+1$ is zero when x is -1, negative for x less than -1, and positive for x greater than -1. The flag going left from -1 shows where $x+1$ is negative. The product $r(x)$ is zero when x is -1 or 3. By counting the negative factors (represented by flags), we can determine the sign pattern of $r(x)$. If $x < -1$, then $r(x)$ is the product of two negative factors, so $r(x)$ is positive. If $-1 < x < 3$, then $r(x)$ has only one negative factor, so it is negative. If $x > 3$, then there are no negative factors, so $r(x)$ is positive. This sign pattern is indicated by the pluses and minuses in Figure 1. We refer to Figure 1 as a **sign diagram** for $r(x)$.

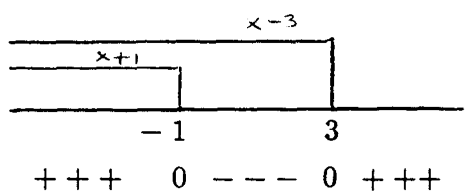

Figure 1. Sign Diagram for $r(x) = (x+1)(x-3)$

Let's next consider

$$r(x) = (2x+1)x^3(x-3)^4.$$

The factor $2x+1$ is zero at $-1/2$, negative to the left of $-1/2$, and positive to the right of $-1/2$. Thus a sign diagram for $r(x)$ should have a flag going left from $-1/2$ (see Figure 2). The factor x^3 is zero at 0, negative to the left of 0, and positive to the right of 0. Figure 2 includes a flag going left from 0 corresponding to this factor. The factor $(x-3)^4$ is zero at 3 and positive everywhere else. Since this factor is never negative, we don't need any corresponding flags. We have, however, erected a flag pole at 3 to remind us that this factor is zero there. Now that we have analyzed the factors, we can count negative factors (flags) to find the sign of $r(x)$. To the left of $-1/2$ there are two negative factors, so $r(x)$ is positive for $x < -1/2$. Between $-1/2$ and 0, there is one negative factor so $r(x)$ is negative for $-1/2 < x < 0$. Between 0 and 3, and to the right of 3, there are no negative factors.

Thus $r(x)$ is positive for $0 < x < 3$ and for $x > 3$. The pluses and minuses in Figure 2 show this sign pattern.

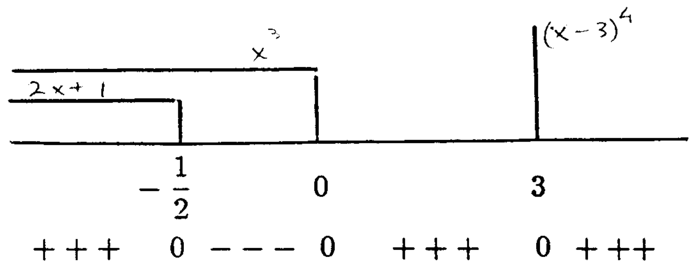

Figure 2. A Sign Diagram for $r(x) = (2x + 1)x^3(x - 3)^4$

One way to analyze the sign of $r(x) = 4 - x^2$ is to factor it as follows:

$$r(x) = -1(x^2 - 4) = -1(x + 2)(x - 2).$$

We then draw a sign diagram (Figure 3) that includes flags indicating where each of the *three* factors of $r(x)$ is negative. Corresponding to the two factors $x + 2$ and $x - 2$, we draw flags going left from -2 and 2. The factor -1 is *always* negative, so we draw a flag over the entire line corresponding to this factor. The number of flags is odd to the left of -2 and to the right of 2. Thus $r(x)$ is negative for $x < -2$ or $x > 2$. Between -2 and 2 there is an even number of flags. Thus $r(x)$ is positive for $-2 < x < 2$.

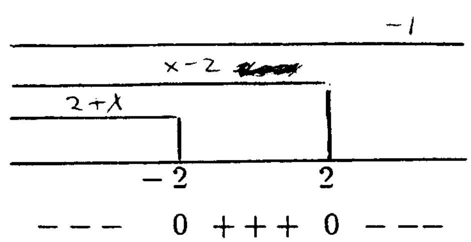

Figure 3. A Sign Diagram for $r(x) = 4 - x^2$

Another way to deal with $r(x) = 4 - x^2$ is to factor it as follows:

$$r(x) = (2 + x)(2 - x).$$

As before, our sign diagram (Figure 4) has a flag going left from -2 showing where $2 + x$ is negative. The factor $2 - x$ is zero at 2, negative for x *greater* than 2, and positive for x *less* than 2. To show where $2 - x$ is negative, we have drawn a flag going *right* from 2. As usual, we count negative factors (flags) to determine the sign pattern of $r(x)$.

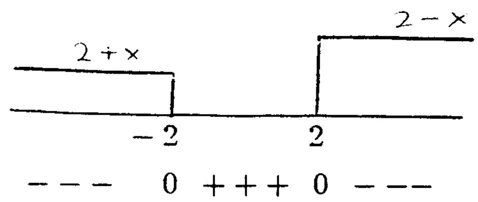

Figure 4. Another Sign Diagram for $r(x) = 4 - x^2$

Another situation that can arise is illustrated by the polynomial

$$r(x) = (3x - 2)(-x^2 + 2x - 5).$$

As usual, our sign diagram (Figure 5) has a flag going left from $2/3$ corresponding to the factor $(3x-2)$. There are no real values of x for which the factor $-x^2+2x-5$ is zero (you can check this by using the quadratic formula). Thus this factor is either always positive or always negative. When x is 0, the factor is -5, so it is always negative. Our sign diagram has a flag over the entire line corresponding to this factor. By counting flags, we see that $r(x)$ is positive for $x < 2/3$ and negative for $x > 2/3$.

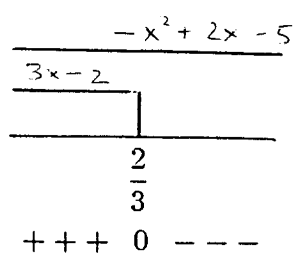

Figure 5. Sign Diagram for $r(x) = (3x-2)(-x^2+2x-5)$

The same method works for quotients. After all, if $x \neq a$, then the sign of $1/(x-a)$ is the same as the sign of $x - a$. Note, however, that $1/(x-a)$ is not defined when $x = a$. We indicate this by placing the symbol $\times$ below $x = a$ on the sign diagram.

As an example, let's analyze the sign of

$$r(x) = \frac{(x^2-3)}{(x+1)^3(x-1)^2} = \frac{(x+\sqrt{3})(x-\sqrt{3})}{(x+1)^3(x-1)^2}.$$

The sign diagram shown in Figure 6 includes flags going left from $\pm\sqrt{3}$ corresponding to the two factors of the numerator. The flag going left from -1 corresponds to the factor $(x+1)^3$ in the denominator. We have erected at flag pole at 1, the point where the factor $(x-1)^2$ is zero. Since this factor is positive everywhere else, we don't need any flags on this pole. The function $r(x)$ is zero when the numerator is zero (at $\pm\sqrt{3}$) and undefined when the denominator is zero (at ± 1). By counting negative factors (flags), we see that $r(x)$ is negative if $x < -\sqrt{3}$, $-1 < x < 1$, or $1 < x < \sqrt{3}$, and that $r(x)$ is positive if $-\sqrt{3} < x < -1$ or $\sqrt{3} < x$.

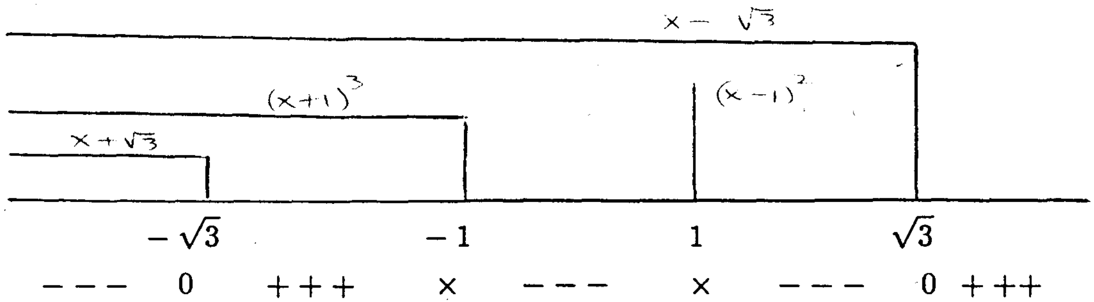

Figure 6. Sign Diagram for $r(x) = \frac{(x^2-3)}{(x+1)^3(x-1)^2}$

As a final example, let's determine the values of x for which the inequality

$$\frac{1}{x+2} \geq \frac{2}{x-1}$$

holds. We first subtract $1/(x-1)$ from both sides of the inequality:

$$\frac{1}{x+2} - \frac{2}{x-1} \geq 0.$$

We next combine the terms on the left side over a common denominator:

$$\frac{-(x+5)}{(x+2)(x-1)} \geq 0.$$

Note that the quotient on the left is zero at -5 and is undefined -2 and at 1. Figure 7 shows a sign diagram for this quotient. From this diagram we see that the inequality holds if

$$x \leq -5 \quad \text{or} \quad -2 < x < 1.$$

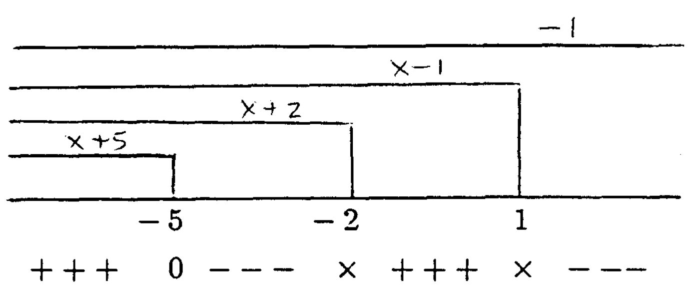

Figure 7. Sign Diagram for $\frac{-(x+5)}{(x+2)(x-1)}$

Exercises

In problems 1 through 9, find the values of x for which $r(x)$ is (a) zero, (b) undefined, (c) positive, and (d) negative.

1. $r(x) = 3x^2 + 5x - 2$
2. $r(x) = 3x^2 - 2x - 4$
3. $r(x) = 2 + x - x^2$
4. $r(x) = 2x^3 + 5x^2 - 3x$
5. $r(x) = x^5 - x$
6. $r(x) = 16 - x^4$
7. $r(x) = \frac{x^3 - 9x}{x^2 - 4}$
8. $r(x) = \frac{-2(x^2+3)(x+2)}{(x-1)^2(x+3)}$
9. $r(x) = \frac{(3x-1)^4(x-3)}{x(x+3)^3}$

In problems 10 through 14, find the values of x for which the given inequality holds.

10. $2x^2 + x - 6 \leq 0$
11. $\frac{x^2-1}{(x-2)^2} \geq 0$
12. $\frac{(x-1)^8}{x^2-25} \geq 0$
13. $x^2 - 3x \leq 3 - x$
14. $\frac{x^2}{x^2-2x+3} \geq 1$

In problems 15 through 17, find the domain of $f(x)$.

15. $f(x) = \sqrt{2 - 3x - 2x^2}$
16. $f(x) = \sqrt{\frac{x^2(x+1)^7}{(x-1)^2(x-2)^3}}$
17. $f(x) = \sqrt{\frac{2+x-x^2}{2x+x^2+x^3}}$