

Summary page: Solutions to the ODE's for the dynamics of springs.

Spring with mass m and stiffness constant k .

Solving $(mD^2 + k)x = 0$, with $m > 0$ and $k > 0$, should be pretty much automatic for us now. The roots of the corresponding characteristic polynomial are $\pm i\sqrt{\frac{k}{m}}$. Hence, the general solution for the displacement from the relaxed position is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where the system frequency is $\omega = \sqrt{\frac{k}{m}}$. Differentiating yields the equation $\dot{x}(t) = v(t) = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t$

for the velocity. Given initial conditions $x(0) = x_0$ and $v(0) = v_0$, the equation for the displacement becomes

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

(Note: It is interesting to work out the units for the equation of a spring. In the gcs system, the mass m is in grams, the displacement x is in centimeters, and time is in seconds. The spring constant k is in g/s^2 which gives kx units of force or dynes in the gcs system.)

In the phase-amplitude form the above solution is $x(t) = A \cos(\omega t - \alpha)$ where $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and α is the angle that satisfies $\cos \alpha = \frac{x_0}{A}$ and $\sin \alpha = \frac{v_0}{\omega A}$ with $0 \leq \alpha < 2\pi$. These formulas for A and α are derived by rewriting $x(t) = A \cos(\omega t - \alpha)$ as $x(t) = A \cos \omega t \cos \alpha + A \sin \omega t \sin \alpha$ and setting $A \cos \alpha = x_0$ and $A \sin \alpha = \frac{v_0}{\omega}$.

Spring with mass m , stiffness constant k , and damping coefficient b .

In the same way that we know how to solve the ODE for the undamped spring, we can write down the general solution to the damped spring, which is governed by the equation $(mD^2 + bD + k)x = 0$. However, there is a significant distinction between the two mechanical systems. On the one hand, the characteristic polynomial for the undamped spring always has pure imaginary roots as described above. On the other hand, the **discriminant** $b^2 - 4mk$ determines the nature of the roots of the corresponding characteristic polynomial $P(r) = mr^2 + br + k$ for the damped spring. The following table contains the details for complex, repeated real, and distinct real roots where, for the specific solution, $x(0) = x_0$ and $v(0) = v_0$:

Discriminant	Stability	System Frequency	Time Constant	General Solution, Specific Solution
< 0	Underdamped	$\omega = \frac{\sqrt{4mk - b^2}}{2m}$ Note that ω is defined in terms of the absolute value of the discriminant.	$\sigma = \frac{b}{2m}$	$x = c_1 e^{-\sigma t} \cos \omega t + c_2 e^{-\sigma t} \sin \omega t$ $x = x_0 e^{-\sigma t} \cos \omega t + \frac{v_0 + \sigma x_0}{\omega} e^{-\sigma t} \sin \omega t$ $x = A e^{-\sigma t} \cos(\omega t - \alpha)$, $A = \sqrt{x_0^2 + \frac{(v_0 + \sigma x_0)^2}{\omega^2}}$, $\cos \alpha = \frac{x_0}{A}$, $\sin \alpha = \frac{v_0 + \sigma x_0}{\omega A}$, $0 \leq \alpha < 2\pi$
$= 0$	Critically Damped	No oscillations (repeated real root)	$\sigma = \frac{b}{2m}$	$x = c_1 e^{-\sigma t} + c_2 t e^{-\sigma t}$ $x = x_0 e^{-\sigma t} + (v_0 + \sigma x_0) t e^{-\sigma t}$
> 0	Overdamped	No oscillations (two real roots)	$\sigma_1 = \frac{b - \sqrt{b^2 - 4mk}}{2m}$ $\sigma_2 = \frac{b + \sqrt{b^2 - 4mk}}{2m}$	$x = c_1 e^{-\sigma_1 t} + c_2 e^{-\sigma_2 t}$ $x = \frac{\sigma_2 x_0 + v_0}{\sigma_2 - \sigma_1} e^{-\sigma_1 t} + \frac{\sigma_1 x_0 + v_0}{\sigma_2 - \sigma_1} e^{-\sigma_2 t}$