CHAPTER

6

LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS: POWER SERIES

6.1 TEMPERATURE MODELS: O.D.E.'S FROM P.D.E.'S

Until now we have focused our attention primarily on linear o.d.e.'s with constant coefficients. In this chapter we develop an approach to solving certain o.d.e.'s with variable coefficients. Although the solutions of these equations may not have finite expressions in terms of elementary functions (polynomials, exponentials, trigonometric functions, and so on), we will be able to find expressions for them in terms of power series. You may recall from calculus that the first several terms of a power series expression can be used to obtain a good approximation to the value of a function at a specified point. (Indeed, if our goal is to obtain a decimal expression for such a value, a series expression can be more useful than a closed-form expression involving exponentials and trigonometric functions.) In addition, series expressions can often be used to obtain powerful information about the behavior of solutions of o.d.e.'s.

Although linear o.d.e.'s with variable coefficients can be obtained by varying our earlier models (see Exercises 1 through 5), in practice they usually arise when we look for special solutions to certain partial differential equations (p.d.e.'s). The way such problems lead to o.d.e.'s is illustrated in this section by several versions of the problem of finding a steady-state (i.e., time-independent) temperature distribution in a body.

In Section 8.1 we discuss more carefully the derivation of p.d.e.'s modeling temperature distribution. For the present, we note that the steady-state temperature \( u(x, y) \) in a two-dimensional plate whose flat surfaces are insulated must satisfy the p.d.e.

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]

(L2)
and the steady-state temperature \( u(x, y, z) \) in a three-dimensional solid satisfies

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
\]

These p.d.e.'s are known as the two- and three-dimensional Laplace equations.

As a first step toward solving these p.d.e.'s, we look for solutions that can be written as products of functions of one variable. Our hope is that we can replace the p.d.e. by o.d.e.'s for these functions. In the following example we illustrate how the search for such special solutions to the two-dimensional Laplace equation can lead to familiar o.d.e.'s. When we return to this problem in Chapter 8, we will see how the solutions to these o.d.e.'s can be combined to find solutions of the Laplace equation that match given boundary conditions.

### Example 6.1.1 Temperature in a Rectangular Plate

Suppose the temperature in a thin rectangular plate, insulated above and below, is controlled by heating (or cooling) elements along its edges (see Figure 6.1). We wish to predict the steady-state temperature, \( u(x, y) \), that satisfies the two-dimensional Laplace equation \((L_2)\) inside the rectangle.

We look for solutions of \((L_2)\) in the special form

\[
u = X(x)Y(y).
\]

Substituting \((S)\) into \((L_2)\), we obtain

\[
\frac{d^2X}{dx^2} \frac{d^2Y}{dy^2} + X \frac{d^2Y}{dy^2} = 0
\]

or

\[
\frac{1}{X} \frac{d^2X}{dx^2} = \frac{1}{Y} \frac{d^2Y}{dy^2}.
\]

Note that the left side of this equation is a function of \(x\) alone, but the right is independent of \(x\). The only way this can happen is if both sides have a common constant value,

\[
\frac{1}{X} \frac{d^2X}{dx^2} = \frac{1}{Y} \frac{d^2Y}{dy^2} = \lambda.
\]

When we write these equations separately, we obtain two ordinary differential equations:

\[
\frac{d^2X}{dx^2} - \lambda X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = \lambda.
\]

These can be rewritten in the more familiar forms

\[
(H_x) \quad [D^2 - \lambda]X = 0 \quad \text{and} \quad (H_y) \quad [D^2 + \lambda]Y = 0
\]

where \(D\) stands for \(d/dx\) in \((H_x)\) and for \(d/dy\) in \((H_y)\).

Note that the solutions of \((H_x)\) and \((H_y)\) depend on the specific values of \(\lambda\); these in turn are determined by the temperature along the edges of the plate (see, for example, Exercise 6). Once we solve \((H_x)\) and \((H_y)\), we substitute into \((S)\) to obtain solutions to \((L_2)\).

In the next examples we will see how o.d.e.'s with variable coefficients arise when we try this procedure on the Laplace equation in polar, cylindrical, or spherical coordinates.

### Example 6.1.2 Temperature in a Circular Plate

To investigate the steady-state temperature distribution in a circular plate of radius \(r\) (insulated above and below) when we know the temperature on the boundary, we are led to the two-dimensional Laplace equation \((L_2)\) in the interior of the disc \(x^2 + y^2 < 1\). For such problems, it is most convenient to switch to polar coordinates. Rewritten in terms of \(r\) and \(\theta\), \((L_2)\) becomes
(Exercise 9)

\[ \frac{\partial^3 u}{\partial r^3} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \]

We look for solutions of (H) in the form

\[ u = R(r) \Theta(\theta). \]

Substituting (S) into (H), we obtain

\[ \frac{\partial^2 R}{\partial r^2} \Theta + \frac{1}{r} \frac{\partial R}{\partial r} \Theta + \frac{1}{r^2} R \frac{\partial^2 \Theta}{\partial \theta^2} = 0. \]

Subtracting the last term from both sides, multiplying by \( r^2 \), and dividing by \( u = R \Theta \), we come to

\[ r^2 \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + \frac{1}{r} \frac{\partial^2 \Theta}{\partial \theta^2} = 0. \]

Since the left side of this equation is a function of \( r \) alone and the right side depends only on \( \theta \), the common value must be a constant, \( \lambda \). This leads to two o.d.e.'s

\[ \frac{\partial^2 \Theta}{\partial \theta^2} = \lambda \]

\[ \frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + \lambda = 0 \]

where \( D \) stands for \( \frac{d}{d\theta} \) in (Hd) and for \( \frac{d}{dr} \) in (Hc).

The first of these equations can be rewritten as

\[ (D^2 + \lambda) \Theta = 0. \]

Since \( \Theta \) is a function of the angular coordinate \( \theta \), it must satisfy the geometric condition \( \Theta(\theta + 2\pi) = \Theta(\theta) \) for all \( \theta \). If this is to hold for all solutions of (Hd), then \( \lambda \) must be a perfect square (Exercise 7):

\[ \lambda = n^2, \quad n \geq 0 \text{ an integer}. \]

Of course, the value of \( \lambda \) in (Hc) must agree with that in (Hd). Thus, we are led to the o.d.e. for \( R(r) \),

\[ \frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} = n^2. \]

Multiplying by \( R \) and rearranging terms, we obtain

\[ \frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + \frac{1}{r^2} R \frac{\partial^2 \Theta}{\partial \theta^2} = 0. \]

This is a special case of the Cauchy-Euler equation

\[ \frac{d^2 x}{dt^2} + q \frac{dx}{dt} + r x = 0 \]

(with \( p = q = 1 \) and \( r = -n^2 \)). We will learn to solve (CE) in Section 6.5.

Example 6.1.3 Temperature in a Plug

A metal plug in the shape of a (solid) cylinder is insulated along the surface; the temperature is controlled by two rings on the top and bottom edges (see Figure 6.2). We wish to predict the steady-state temperature in the plug, \( u(x, y, z) \), which must satisfy the three-dimensional Laplace equation (L3).

\[ \text{FIGURE 6.2} \]

We start by looking for solutions of (L3) in the form

\[ u = U(x, y)Z(z). \]

Substituting (S1) into (L3), we get

\[ \frac{\partial^2 U}{\partial x^2} Z + \frac{\partial^2 U}{\partial y^2} Z + U \frac{\partial^2 Z}{\partial z^2} = 0 \]

or

\[ \frac{1}{U} \left[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right] = \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} \]

Since the right side is a function of \( z \) alone and the left side is independent of
z, each is constant, and we obtain two equations

\[ \frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda \]

\[ \frac{1}{U} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = \lambda. \]

The first equation is again the familiar o.d.e.

\[ [D^2 + \lambda]Z = 0, \]

(where \( D = d/dz \)), whose solutions depend on \( \lambda \). It turns out that physical considerations lead to the requirement that \( \lambda \) is nonpositive, so

\[ \lambda = -\gamma^2 \]

for some \( \gamma \geq 0 \).

The shape of the plug leads us to try to rewrite the function \( U \) in terms of polar coordinates \( r \) and \( \theta \). The p.d.e. \( (H_0) \) now takes the form (Exercise 9)

\[ \frac{1}{U} \left( \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right) = -\gamma^2. \]

When \( \gamma = 0 \), this is equivalent to the p.d.e. of the previous example. When \( \gamma > 0 \), we can still look for solutions to \( (H_0) \) in the separated form

\[ U = R(r)\Theta(\theta). \]

Substituting \( S_2 \) into \( (H_0) \), we obtain

\[ \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{d R}{d r} + \frac{1}{r^2 \Theta} \frac{d^2 \Theta}{d \theta^2} = -\gamma^2 \]

or, multiplying by \( r^2 \) and rearranging terms,

\[ r^2 \frac{d^2 R}{dr^2} + \frac{r d R}{d r} + (\gamma r)^2 - \frac{1}{\Theta} \frac{d \Theta}{d \theta^2} = 0. \]

Once again, the two sides of this equation depend on different variables, so we set each equal to a common constant, \( \kappa \):

\[ \frac{-1}{\Theta} \frac{d^2 \Theta}{d \theta^2} = \kappa \]

\[ \frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{r}{R} \frac{d R}{d r} + (\gamma r)^2 = \kappa. \]

As in Example 6.1.2, the geometric condition \( \Theta(\theta + 2\pi) = \Theta(\theta) \) forces \( \kappa \) to be a perfect square:

\[ \kappa = n^2, \quad n \geq 0 \text{ an integer}. \]

The substitution \( t = \gamma r \) then puts \( (H_0) \) in the form of the Bessel equation

\[ [r^2 D^2 + tD + (r^2 - \mu^2)]x = 0 \]

with \( D = d/dt, \ x = R, \) and \( \mu = n \). We will discuss this equation in Sections 6.7 and 6.8.

Example 6.1.4 Temperature in a Planet

Suppose the temperature at the surface of a spherical planet is a function of the latitude alone (see Figure 6.3). We would like to predict the internal temperature distribution \( u \), which we assume to be steady-state.

![Isotherms](image)

**FIGURE 6.3**

Again, \( u \) must satisfy the three-dimensional Laplace equation \( (L_3) \), but here we expect it to be most convenient to express \( u \) in terms of spherical coordinates \( \rho, \phi, \) and \( \theta \) (see Figure 6.4). We have assumed that the temperature on the surface depends on the latitude \( \phi \) and not on the longitude \( \theta \), so we expect the same to be true inside the planet. When \( u = u(\rho, \phi) \) is independent of \( \theta \), the three-dimensional Laplace equation becomes (Exercise 10)

\[ \frac{\rho^2}{\sin \phi} \frac{\partial^2 (\rho u)}{\partial \phi^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) = 0. \]

We look for a solution of \( (H) \) in the separated form

\[ u = R(\rho)\Phi(\phi). \]
we make the substitution \( \phi = \cos \theta \), we obtain (Exercise 8)

\[
\frac{1}{\Phi} \frac{d}{dt} \left[ (t^2 - 1) \frac{d\Phi}{dt} \right] = n(n + 1).
\]

After some manipulation, this takes the form of the Legendre equation

\[
[(t^2 - 1)D^2 + 2tD - n(n + 1)]\Phi = 0.
\]

**EXERCISES**

1. **A Circuit with Varying Resistance**: The resistance of metals varies with temperature. Suppose the resistor of Exercise 5, Section 2.1 (or Example 3.1.1) is heated so that its resistance at time \( t \) is \( R(t) = 10 + t \). Find an o.d.e. for the charge \( Q \) on the capacitor if \( C = 1 \) and \( V(t) = 10 \).

2. **A Varying Spring Constant**: When a spring is heated, the spring constant may change. Write an o.d.e. modeling the spring system of Example 2.1.2, with \( m = b = 10 \), under the assumption that the spring "constant" at time \( t \) is \( k(t) = 10 - t \).

3. **Varying Damping**: As oil is heated, it becomes slipperier. Write an o.d.e. modeling the spring system of Example 2.1.2, with \( m = k = 10 \), under the assumption that the spring is attached to a dashpot that is being heated so that the damping coefficient at time \( t \) is \( b(t) = 10 - t \).

4. **Varying Mass**: Suppose the mass attached to the spring of Example 2.1.2 is a 10 gram block of ice that melts at the rate of 1 gram per second. Write an o.d.e. modeling the motion of the system if \( k = 10 \) and \( b = 11 \). [Note: When mass varies, Newton's second law takes the form \( F = d(mv)/dt \). Use this, not \( F = ma \), as the basis for this model.]

5. **A Submarine**: Recall the principle of Archimedes (Exercise 7, Section 2.1), which states that water buoys up a submerged object by a force equal to the weight of the water it displaces. Suppose a submarine weighing 50 tons (100,000 lb) is resting on the ocean floor and that the buoyant force exactly counters the force of gravity at this stage. Now, water is pumped out of its tanks at 100 lb/second.
   a. Write a formula for the mass \( m(t) \) of the submarine after \( t \) seconds.
   b. Write a formula for the net force \( F(t) \) acting on the submarine after \( t \) seconds.
   c. Use Newton's second law in the form for varying mass \( F = d(mv)/dt \), not \( F = ma \) to derive a second-order o.d.e. modeling the position of the submarine, assuming no damping forces.

6. **The Rectangular Plate**: Suppose the temperatures of the bottom edge \( y = 0 \) and the top edge \( y = h \) of the plate in Example 6.1.1 are held at \( 0^\circ \); that is, 

\[
u(x, 0) = u(x, h) = 0 \quad \text{for all } x.
\]
a. Show that if \( u = X(x) Y(y) \) satisfies (B), but is not identically zero, then
\[
Y(0) = Y(h) = 0.
\]

b. Show that if \( \lambda < 0 \), then the only solution of (H₄) that satisfies (B₁) is the trivial solution \( Y = 0 \).

c. Show that if \( \lambda = 0 \), then (H₄) has no nontrivial solutions that satisfy (B₁).

d. Show that if \( \lambda > 0 \), then (H₄) has nontrivial solutions that satisfy (B₁) only if
\[
\sin h \sqrt{-\lambda} = \frac{1}{n},
\]
where \( n \) is a nonnegative integer.

e. Show that if \( \lambda = n^2 \pi^2 h^2 \), then the nontrivial solutions of (H₄) that satisfy (B₁) are constant multiples of
\[
y = \sin \left( \frac{n \pi y}{h} \right),
\]
and (H₄) has two linearly independent solutions,
\[
xₙ = e^{n \pi x h} \quad \text{and} \quad xₙ = e^{-n \pi x h}.
\]

f. Corresponding to the solutions of (H₄) and (H₅) described in (e), obtain solutions to (L₂),
\[
uₙ = e^{-n \pi x h} \sin \left( \frac{n \pi y}{h} \right) \quad \text{and} \quad uₙ = e^{-n \pi x h} \cos \left( \frac{n \pi y}{h} \right)
\]
that satisfy (B). Note that any linear combination of these (infinitely many) functions is a solution of (L₂) and satisfies (B).

Exercises 7 through 10 fill some of the gaps in the discussion of Examples 6.12 through 6.14.

7. a. Show that if \( \lambda < 0 \), then the only solution of
\[
(H₄) \quad \frac{d^2 \Theta}{d\theta^2} + \lambda \Theta = 0.
\]
that satisfies the geometric condition
\[
\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for all} \ \theta
\]
is \( \Theta = 0 \).

b. Show that if \( \lambda = 0 \), then the only solutions of (H₄) that satisfy (G) are constant multiples of \( \Theta = 1 \).

c. Show that if \( \lambda > 0 \), then (H₄) has nontrivial solutions that satisfy (G) only if
\[
\lambda = n^2, \quad \text{where} \ n \text{ is a nonnegative integer}.
\]

8. a. Show that the substitution \( t = \cos \phi \) changes the equation
\[
(H₄) \quad \frac{1}{\Phi \sin \phi} \frac{d}{d\phi} \left[ \sin \phi \frac{d \Phi}{d\phi} \right] = n(n + 1)
\]
into
\[
\frac{1}{\Phi} \frac{d}{dt} \left[ (t^2 - 1) \frac{d \Phi}{dt} \right] = n(n + 1).
\]

b. Show that this last equation can be rewritten in the form
\[
[(t^2 - 1) \frac{d}{dt} + 2D - n(n + 1)] \Phi = 0.
\]

9. Conversion to Polar Coordinates

a. Use the chain rule for partial derivatives to show that if \( v = v(x, y) \), where \( x = r \cos \theta \) and \( y = r \sin \theta \), then
\[
\frac{\partial v}{\partial r} = \cos \theta \frac{\partial v}{\partial x} + \sin \theta \frac{\partial v}{\partial y},
\]
\[
\frac{\partial v}{\partial \theta} = -r \sin \theta \frac{\partial v}{\partial x} + r \cos \theta \frac{\partial v}{\partial y}.
\]

b. Solve the preceding equations for \( \frac{\partial v}{\partial r} \) and \( \frac{\partial v}{\partial \theta} \) in terms of \( \frac{\partial v}{\partial x} \) and \( \frac{\partial v}{\partial y} \),
\[
\frac{\partial v}{\partial x} = \cos \theta \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta},
\]
\[
\frac{\partial v}{\partial y} = \sin \theta \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta}
\]

\[c. \text{By applying the equations in (b), first to} \ v = u(x, y) \text{and then to} \ v = \partial u/\partial x \text{and} \ v = \partial u/\partial y, \text{show that}
\]
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\]

\[d. \text{Conclude that the change of variables} \ x = r \cos \theta \text{and} \ y = r \sin \theta \text{changes}
\]
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \beta
\]
into
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \beta.
\]

*10. Conversion to Spherical Coordinates

a. Suppose \( v = v(x, y, z) \), where \( x = r \sin \phi \cos \theta \), \( y = r \sin \phi \sin \theta \), and \( z = r \cos \phi \). Use the chain rule to express \( \partial v/\partial \phi, \partial v/\partial \theta, \) and \( \partial v/\partial r \) in terms of \( \partial v/\partial x, \partial v/\partial y, \) and \( \partial v/\partial z \).

b. Solve the equations in (a) for \( \partial v/\partial r, \partial v/\partial \phi, \) and \( \partial v/\partial \theta \) in terms of \( \partial v/\partial x, \partial v/\partial y, \) and \( \partial v/\partial z \).

\[c. \text{By applying the equations in (b) to} \ v = u(x, y, z) = \partial u/\partial x = \partial u/\partial y = \partial u/\partial z \text{, show that}
\]
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\rho^2}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho} \frac{\partial u}{\partial \phi} \left( \sin \phi \frac{\partial \phi}{\partial \phi} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial^2 u}{\partial \phi^2}.
\]

\[d. \text{Conclude that if} \ u \text{is independent of} \ \phi, \text{then the change of variables} \ x = r \sin \phi \cos \theta, \ y = r \sin \phi \sin \theta, \text{and} \ z = r \cos \phi \text{changes the three-dimensional Laplace equation (L₂) into}
\]
\[
\rho^2 \frac{\partial^2 u}{\partial \rho^2} + \frac{\partial u}{\partial \rho} = 0.
\]
6.2 REVIEW OF POWER SERIES

In this section we recall some features of power series that we will find useful in solving o.d.e.'s. Our review is brief; you can find a more elaborate treatment in most good calculus texts.

A series is a formal sum

\[ \sum_{k=0}^{\infty} u_k = u_0 + u_1 + \cdots + u_k + \cdots. \]

To evaluate this series, we consider the partial sums

\[ S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + \cdots + u_n. \]

If the partial sums approach a limit

\[ S = \lim_{n \to \infty} S_n, \]

we call \( S \) the sum of the series,

\[ \sum_{k=0}^{\infty} u_k = S \]

and say the series converges to \( S \). If the partial sums do not approach a limit, we say the series diverges.

There are many tests for determining whether a series converges. The most useful one for our purposes is the ratio test.

**Fact: Ratio Test.** Given the series \( \sum_{k=0}^{\infty} u_k \), suppose that

\[ \rho = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|. \]

Then:

1. If \( \rho < 1 \), the series converges.
2. If \( \rho > 1 \), the series diverges.
3. If \( \rho = 1 \), the test fails. The series may converge or diverge.

We will be dealing with power series. A power series about \( t = t_0 \) is a series of the form

\[ \sum_{k=0}^{\infty} b_k (t - t_0)^k - b_0 + b_1 (t - t_0) + \cdots + \sqrt[k]{b_k (t - t_0)^k + \cdots}, \]

where the \( b_k \)'s and \( t_0 \) are constants, and \( t \) is a variable. Note that the partial sums of a power series are polynomials. The basic convergence properties of power series are given by the following statement.

**Fact:** Associated to each power series \( \sum_{k=0}^{\infty} b_k (t - t_0)^k \) is a "number" \( R \), with \( 0 \leq R \leq \infty \), so that

1. If \( 0 < R < \infty \), the series converges whenever \( |t - t_0| < R \) and diverges whenever \( |t - t_0| > R \);
2. If \( R = \infty \), the series converges for all \( t \); and
3. If \( R = 0 \), the series converges only when \( t = t_0 \).

The "number" \( R \) is called the *radius of convergence* of the power series. In cases 1 and 2, when \( R > 0 \), we refer to the values of \( t \) with \( |t - t_0| < R \) (that is, \( t_0 - R < t < t_0 + R \)) as the interval of convergence of the power series. Note that in case 1 nothing was said about the endpoints \( (t = t_0 \pm R) \) of this interval; the series may converge or diverge at either point.

The radius of convergence of a power series can usually be calculated by means of the ratio test.

**Example 6.2.1**

Find the interval of convergence of \( \sum_{k=0}^{\infty} t^k \).

We apply the ratio test. Since

\[ \rho = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{t^{n+1}}{t^n} \right| = \lim_{n \to \infty} |t| = |t|, \]

the series converges for \( |t| < 1 \) and diverges for \( |t| > 1 \). The interval of convergence is \( |t| < 1 \).

**Example 6.2.2**

Find the interval of convergence of \( \sum_{k=0}^{\infty} (t^k/k!) \).

By the ratio test, since

\[ \rho = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{t^{n+1}(n+1)!}{t^n n!} \right| \]

\[ = \lim_{n \to \infty} \frac{t^{n+1}(n+1)!}{(n+1)! |t^n/ (n+1)!|} = \lim_{n \to \infty} \left| \frac{t}{n+1} \right| = 0 < 1, \]
the series always converges. The interval of convergence is the whole real line, $|t| < \infty$.

If a power series has $R > 0$, then it defines a function on its interval of convergence. Conversely, given a function $f(t)$, it may be possible to find a power series about $t = t_0$ so that

$$f(t) = \sum_{k=0}^{\infty} b_k(t - t_0)^k \text{ for } |t - t_0| < R,$$

where $R > 0$. If such a series exists, we say that $f(t)$ is analytic at $t = t_0$.

As examples, we list some basic functions that are analytic at $t = 0$, together with expressions for these functions as series. (See Examples 6.2.4 and 6.2.6, and Exercise 31.)

**Fact:** The following functions are analytic at $t = 0$.

$$b_0 + b_1 t + \cdots + b_m t^m = b_0 + b_1 t + \cdots + b_m t + 0 t^{m+1} + \cdots, \quad |t| < \infty$$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \quad |t| < \infty$$

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}, \quad |t| < \infty$$

$$\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}, \quad |t| < \infty$$

$$\frac{1}{1 - t} = \sum_{k=0}^{\infty} t^k, \quad |t| < 1.$$

In this chapter, we look for analytic solutions of an o.d.e. by substituting a formal power series $\Sigma_{k=0}^{\infty} b_k(t - t_0)^k$ into the equation and attempting to determine the constants $b_k$. In order to carry out this process, we need to know when a power series is identically equal to zero, how to do arithmetic with power series, and how to differentiate power series. Equality of a series with zero is governed by the following (see Exercise 32):

**Fact:** If $\Sigma_{k=0}^{\infty} b_k(t - t_0)^k = 0$ for all $t$ satisfying $|t - t_0| < R$, where $R > 0$, then $b_k = 0$ for every $k$.

The basic arithmetic operations for power series at $t = t_0$ are analogous to those for polynomials. In particular, we multiply a series by a number or add two series, termwise.

**Fact:** If the series expressions

$$f(t) = \sum_{k=0}^{\infty} b_k(t - t_0)^k \quad \text{and} \quad g(t) = \sum_{k=0}^{\infty} c_k(t - t_0)^k$$

are both valid (at least) for $|t - t_0| < R$, where $R > 0$, then so are the expressions

$$af(t) = \sum_{k=0}^{\infty} (ab_k)(t - t_0)^k$$

$$f(t) + g(t) = \sum_{k=0}^{\infty} (b_k + c_k)(t - t_0)^k.$$

The multiplication and division of series is more complicated (see Notes 1 and 2); the multiplication of a power series by a polynomial is illustrated in the following examples.

### Example 6.2.3

Find the terms up to $t^4$ in the expression for

$$e^{2t} - (2 + t) \cos t$$

as a power series about $t = 0$.

The series expression for $e^t$ described above is valid for all values of $t$. We obtain the expression for $e^{2t}$ by replacing $t$ with $2t$:

$$e^{2t} = 1 + 2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{6} + \frac{(2t)^4}{24} + \cdots$$

$$= 1 + 2t + 2t^2 + \frac{4}{3} t^3 + \frac{2}{3} t^4 + \cdots.$$

Next, we multiply the series for $\cos t$ by $(2 + t)$:

$$(2 + t) \cos t = (2 + t) \left[ 1 + \frac{t^2}{2} + \frac{t^4}{24} + \cdots \right]$$

$$= (2 + t) \left( \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{120} + \cdots \right).$$
Finally, we subtract the expression for \((2 + t)\cos t\) from the expression for \(e^{2t} - (2 + t)\cos t = -1 + t + 3t^2 + \frac{11}{6} t^3 + \frac{7}{12} t^4 + \cdots \). 

**Example 6.2.4**

Verify that

\[
\frac{1}{1 - t} = \sum_{k=0}^{\infty} t^k \quad \text{for } |t| < 1.
\]

We multiply the series on the right by \((1 - t)\); since this series converges only for \(|t| < 1\) (see Example 6.2.1), we must restrict our attention to this interval:

\[
(1 - t) \sum_{k=0}^{\infty} t^k = \sum_{k=0}^{\infty} t^k - t \sum_{k=0}^{\infty} t^k = \sum_{k=0}^{\infty} t^k - \sum_{k=1}^{\infty} t^{k+1} = \sum_{k=0}^{\infty} t^k - t = \sum_{k=1}^{\infty} t^k, \quad |t| < 1.
\]

To perform our subtraction termwise, we must match up the powers of \(t\). We accomplish this by index substitution—substitute \(j = k\) in the first series and \(j = k + 1\) in the second, remembering that this also affects the limits of summation:

\[
(1 - t) \sum_{k=0}^{\infty} t^k = \sum_{j=0}^{\infty} t^j - \sum_{j=1}^{\infty} t^j = \sum_{j=0}^{\infty} t^j - \sum_{j=1}^{\infty} t^j = \sum_{j=0}^{\infty} t^j - t = \sum_{j=1}^{\infty} t^j, \quad |t| < 1.
\]

Now we have to match up the limits of summation. We do this by breaking the first sum in two:

\[
(1 - t) \sum_{k=0}^{\infty} t^k = \left( \sum_{j=0}^{\infty} t^j \right) - \sum_{j=1}^{\infty} t^j = 1, \quad |t| < 1.
\]

Thus, dividing both sides by \(1 - t\), we obtain the desired equality:

\[
\sum_{k=0}^{\infty} t^k = \frac{1}{1 - t}, \quad |t| < 1.
\]

**Example 6.2.5**

Find a series expression for \(1/(1 - t)^2\) valid for \(|t| < 1\).

We know that

\[
\frac{1}{1 - t} = \sum_{k=0}^{\infty} t^k, \quad |t| < 1.
\]

Differentiation gives

\[
\frac{1}{(1 - t)^2} = \sum_{k=1}^{\infty} kt^{k-1}, \quad |t| < 1.
\]

The limits of summation can be changed by the index substitution \(j = k - 1\) (and \(k = j + 1\)):

\[
\frac{1}{(1 - t)^2} = \sum_{j=0}^{\infty} (j + 1)t^j, \quad |t| < 1.
\]
Example 6.2.6

Verify that

\[ e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \text{ for all } t. \]

Denote the series on the right side by \( f(t) \), recalling (Example 6.2.2) that it converges for all \( t \):

\[ f(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \quad |t| < \infty. \]

Differentiation yields

\[ f'(t) = \sum_{k=1}^{\infty} \frac{k t^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{(k-1)!}{k!} t^{k-1} \]

and the index substitution \( j = k - 1 \) (or \( k = j + 1 \)) gives

\[ f'(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} = f(t), \quad |t| < \infty. \]

Thus \( x = f(t) \) is a solution of

\[ (D - 1)x = 0. \]

Then

\[ f(t) = ce^t, \]

and since \( f(0) = 1 \), we have \( c = 1 \). It follows that \( f(t) = e^t \), or

\[ \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t, \quad |t| < \infty. \]

Our rules for manipulating series enable us to find power series expressions for a wide variety of functions, starting from our expressions for \( e^t \), \( \sin t \), \( \cos t \), and \( \frac{1}{1-t} \). We discuss briefly the problem of finding power series expressions for more complicated functions in the notes following our summary.
division. For example, we can find the first terms of the series for \(1/(1 - t)\) as follows:

\[
1 - t \quad \frac{1}{1 - t} \quad \frac{1}{t} \quad \frac{t^2}{t^2 - t^3} \quad \frac{t^3}{t^3 - t^4} \quad \cdots
\]

3. Taylor series

Suppose we are given a function \(f(t)\) for which we want to find a power series expansion about \(t = t_0\). If there is such an expression

\[
f(t) = \sum_{k=0}^{\infty} b_k(t - t_0)^k
\]

valid for \(|t - t_0| < R\), where \(R > 0\), then

\[
f^{(k)}(t_0) = k!b_k,
\]

so the coefficients must be given by

\[
b_k = \frac{f^{(k)}(t_0)}{k!}.
\]

The series with these coefficients,

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} (t - t_0)^k
\]

is known as the Taylor series for \(f(t)\) about \(t = t_0\). It is the only series about \(t = t_0\) that could possibly equal \(f(t)\).

In practice, it may be easy to decide where the Taylor series converges but difficult to determine whether its limit equals \(f(t)\). Indeed, there are examples of functions \(f(t)\) whose Taylor series converge, but not to \(f(t)\).

Examples 6.2.4 to 6.2.6 illustrated some of the methods used to show that a function is equal to a given series. Another method is to study the “remainder term”

\[
R_k(t, t_0) = f(t) - \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} (t - t_0)^k.
\]
The function $f(t)$ equals its Taylor series precisely when
\[
\lim_{n \to \infty} R_n(t, t_0) = 0.
\]

**EXERCISES**

In Exercises 1 through 9, find the interval of convergence.

1. \[\sum_{k=0}^{\infty} \frac{2^k t^k}{k!} \]
2. \[\sum_{k=0}^{\infty} \frac{2^k t^k}{(k + 1)!} \]
3. \[\sum_{k=0}^{\infty} \frac{(-1)^k k^k t^k}{(k + 1)!} \]
4. \[\sum_{k=0}^{\infty} \frac{(-1)^k (k + 1)^k t^k}{k!} \]
5. \[\sum_{k=0}^{\infty} \frac{(2^k + 1)^k t^k}{k!} \]
6. \[\sum_{k=0}^{\infty} \frac{(k + 1)^k t^k}{k!} \]
7. \[\sum_{k=0}^{\infty} \frac{(t - 2)^k}{(3k + 1)!} \]
8. \[\sum_{k=0}^{\infty} \frac{(2t + 1)^k}{k!} \]
9. \[\sum_{k=0}^{\infty} \frac{k!(2t - 1)^k}{(2k + 1)!} \]

In Exercises 10 through 17, perform the indicated formal operation.

10. \[\sum_{k=0}^{\infty} t^k - \sum_{k=0}^{\infty} \frac{t^k}{(k + 1)!} \]
11. \[\sum_{k=0}^{\infty} t^k + \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \]
12. \[\sum_{k=0}^{\infty} t^k + \sum_{k=0}^{\infty} \frac{t^k}{k!} \]
13. \[1 + t \sum_{k=0}^{\infty} (-1)^k t^k \]
14. \[(t^2 - 1) \sum_{k=0}^{\infty} \frac{3^k t^k}{k!} \]
15. \[\frac{d}{dt} \left( \sum_{k=0}^{\infty} \frac{3^k (3k + 1)! t^k}{k!} \right) \]
16. \[\frac{d^2}{dt^2} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (k + 1)!}{k!} t^k \right) \]

In Exercises 18 through 22, find the terms up to $t^2$ of the given product or quotient (see Notes 1 and 2).

18. \[\sum_{k=0}^{\infty} \frac{t^k}{(k + 1)!} \left( \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \right) \]
19. \[\left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \right)^2 \]
20. \[\sum_{k=0}^{\infty} \frac{(-1)^k k^k t^k}{(2k)!} \left( \sum_{k=0}^{\infty} \frac{(2k + 1)!}{k!} t^k \right) \]
21. \[\sum_{k=0}^{\infty} \frac{t^k}{(2k + 1)!} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \]
22. \[\sum_{k=0}^{\infty} \frac{2^k t^k}{k!} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \]

Starting with the basic series expressions for $\sin t$, $\cos t$, $e^t$, and $1/(1 - t)$, find series expressions for the functions in Exercises 23 through 30.

**EXERCISES (Continued)**

23. \[\sin \left( \frac{t}{2} \right) \]
24. \[\cos 2t \]
25. \[e^{-t} \]
26. \[\frac{1}{1 + t} \]
27. \[\frac{1}{(1 + t)^2} \]
28. \[\frac{1}{(1 + t)^3} \]
29. \[\frac{1}{1 + t^2} \]
30. \[\cos^2 t \text{ (Hint: Use a trig identity or see Note 1.)} \]

Some more abstract problems:

31. **Series Expressions for $\sin t$ and $\cos t$:**
   a. Show that $\sum_{k=0}^{\infty} (-1)^k t^{2k+1}/(2k + 1)!$ converges for all $t$.
   b. Verify that $\sin t$ and the series in (a) are equal by showing that they are solutions of the same initial value problem:
   \[(D^2 + 1)x = 0, \quad x(0) = 0, \quad x'(0) = 1.\]
   c. Obtain an expression for $\cos t$ by differentiating the expression for $\sin t$.

32. **Equality of a Series with Zero:** Show that if $f(t) = \sum_{k=0}^{\infty} b_k (t - t_0)^k = 0$ for all $t$ satisfying $|t - t_0| < R$ ($R > 0$), then $b_n = 0$, $k = 0, 1, \ldots$ [Hint: Express $b_k$ in terms of $f^{(k)}(t_0)$.]

33. **Consider the function $f(t) = e^{-t^2}$, for $t \neq 0$ and $f(t) = 0$, for $t = 0$.**
   a. Show that $D(e^{-t^2}) = (2t^2)e^{-t^2}$ for $t \neq 0$.
   b. Show that in general $D^n(e^{-t^2}) = (P_n(t) t^{2n})e^{-t^2}$ for $t \neq 0$, where $P_n(t)$ is a polynomial of degree at most $2(n - 1)$.
   c. Show that $f^{(n)}(0) = 0$ for each $n$. [Hint: Use the fact that for any $\alpha$, $e^{\alpha x} e^{-x^2} \to 0$ as $x \to 0$.]
   d. Show that the Taylor series for $f(t)$ expanded about $t = 0$ (see Note 3) converges for all $t \neq 0$, but the sum does not equal $f(t)$ for $t > 0$.

**6.3 SOLUTIONS ABOUT ORDINARY POINTS**

We will see in this section how to find power series expressions for the solutions to linear o.d.e.'s

\[ (N) \quad [a_n(t)D^n + a_{n-1}(t)D^{n-1} + \cdots + a_0(t)]x = E(t) \]

in case the coefficients $a_n(t), \ldots, a_0(t)$ and the forcing term $E(t)$ are all polynomials. We will find such series by substituting the expression

\[ x(t) = \sum_{k=0}^{\infty} b_k (t - t_0)^k \]
into (N) and solving for the coefficients \( b_k \). This straightforward approach always works when \( t_0 \) is a point at which the leading coefficient does not vanish. Such points (where \( a_n(t_0) \neq 0 \)) are called the ordinary points of (N) to distinguish them from the singular points of (N), where \( a_n(t_0) = 0 \).

The following theorem justifies our search for power series solutions of (N) about an ordinary point. Moreover, it guarantees the validity of the answers we obtain as long as \( r \) is closer to \( t_0 \) than the nearest complex root of \( a_n(t) \).

**Theorem:** Suppose that (N) is a linear nth-order o.d.e., with polynomial coefficients and forcing term and that \( t_0 \) is an ordinary point of (N). If \( a_n(z) \neq 0 \) for every complex number \( z \) satisfying \( |z - t_0| < R \), then any solution of (N) has an expression as a power series about \( t = t_0 \) that is valid for (at least) \( |t - t_0| < R \).

To see how the procedure works in practice, we start with a familiar o.d.e.

**Example 6.3.1**

Find a power series expression about \( t = 0 \) for the general solution of

\[
(D^2 - 1)x = 0.
\]

(H)

Note that (H) has no singular points.

We look for solutions of (H) in the form

\[
x(t) = \sum_{k=0}^{\infty} b_k t^k,
\]

(S)

\[
x'(t) = \sum_{k=1}^{\infty} kb_k t^{k-1}, \quad x''(t) = \sum_{k=2}^{\infty} k(k-1)b_k t^{k-2}
\]

and substitute into (H):

\[
\sum_{k=2}^{\infty} k(k-1)b_k t^{k-2} - \sum_{k=0}^{\infty} b_k t^k = 0.
\]

In order to combine these two series into one, we need to match up powers of \( t \). To this end, we perform the index substitutions \( j = k - 2 \) (or \( k = j + 2 \)) in the first series and \( j = k \) in the second; we must remember to adjust the limits of summation as well as the summands. We get

\[
\sum_{j=0}^{\infty} (j + 2)(j + 1)b_{j+2} t^j - \sum_{j=0}^{\infty} b_j t^j = 0
\]

or

\[
\sum_{j=0}^{\infty} [(j + 2)(j + 1)b_{j+2} - b_j] t^j = 0.
\]

Since a power series is identically zero only if all its coefficients are zero, we conclude that

\[
(j + 2)(j + 1)b_{j+2} - b_j = 0
\]

or

\[
b_{j+2} = \frac{1}{(j + 2)(j + 1)} b_j \quad \text{for } j = 0, 1, \ldots
\]

(R)

An equation such as (R), which relates a given coefficient to earlier ones, is called a recurrence relation.

Although (R) tells us nothing about \( b_0 \) or \( b_1 \), we can use it to find all of the later coefficients in terms of these two. When \( j = 0 \) and \( j = 1 \), respectively, (R) gives

\[
b_2 = \frac{1}{2 \cdot 1} b_0, \quad b_3 = \frac{1}{3 \cdot 2} b_1.
\]

Now these expressions can be substituted back into (R) with \( j = 2 \) and 3, respectively, to find

\[
b_4 = \frac{1}{4 \cdot 3} b_2 = \frac{1}{4 \cdot 3 \cdot 2} b_0, \quad b_5 = \frac{1}{5 \cdot 4} b_3 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} b_1.
\]

Continuing in this way, we can express all even-numbered coefficients as multiples of \( b_0 \) and all odd-numbered coefficients as multiples of \( b_1 \):

\[
b_{2m} = \frac{1}{(2m)!} b_0, \quad b_{2m+1} = \frac{1}{(2m + 1)!} b_1.
\]

We substitute these values into (S) and separate the terms involving \( b_0 \).
from those involving \( b_1 \) to obtain

\[
x(t) = b_0 \sum_{m=0}^{\infty} \frac{1}{(2m)!} t^{2m} + b_1 \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} t^{2m+1}.
\]

Since the leading coefficient of (H) never vanishes, this series expression is valid for all \( t \).

It should not surprise us that our answer involves two arbitrary constants, \( b_0 \) and \( b_1 \), since we are solving a second-order o.d.e. These constants have a simple relation to initial conditions:

\[
x(0) = b_0, \quad x'(0) = b_1.
\]

You should verify that substituting \( b_0 = b_1 = 1 \) (respectively \( b_0 = -b_1 = 1 \)) into (R) and (S) gives a series expression for the solution \( x = e^t \) (respectively \( x = e^{-t} \)), which played an important role in our earlier method for solving (H).

This example illustrates the general pattern. We substitute a formal power series for \( x(t) \) into the o.d.e., combine terms, and obtain a recurrence relation. We then use the recurrence relation to determine the coefficients of the series.

We next consider a variable-coefficient example.

---

**Example 6.3.2**

Solve about \( t = 0 \)

\[(t^2 - 1)D^2 - 2)x = 0.\]

Note that \( t = 0 \) is an ordinary point of (H); the only singular points are \( t = \pm 1 \).

We look for solutions in the form

\[
x(t) = \sum_{k=0}^{\infty} b_k t^k.
\]

Substitution into (H) yields

\[
(t^2 - 1) \sum_{k=2}^{\infty} k(k - 1)b_k t^{k-2} - 2 \sum_{k=0}^{\infty} b_k t^k = 0
\]

or

\[
\sum_{k=2}^{\infty} k(k - 1)b_k t^{k-2} - \sum_{k=0}^{\infty} 2b_k t^k = 0.
\]

To match up the powers of \( t \), we substitute \( j = k - 2 \) (or \( k = j + 2 \)) in the middle series and \( j = k \) in the other two:

\[
\sum_{j=0}^{\infty} j(j - 1)b_j t^j - \sum_{j=0}^{\infty} (j + 2)(j + 1)b_{j+2} t^j - \sum_{j=0}^{\infty} 2b_j t^j = 0.
\]

Now the limits of summation don’t match, so we separate the terms for \( j = 0 \) and \( j = 1 \) from the last two series.

\[
\left[ \sum_{j=2}^{\infty} j(j - 1)b_j t^j \right] - \left[ 2b_2 + 6b_3 t + \sum_{j=3}^{\infty} (j + 2)(j + 1)b_{j+2} t^j \right] = 0.
\]

Combining terms, we have

\[
-(2b_2 + 6b_3 t) + \sum_{j=3}^{\infty} [j(j - 1)b_j - (j + 2)(j + 1)b_{j+2} - 2b_j] t^j = 0.
\]

This can happen only if the coefficient of each power of \( t \) is zero. Thus

\[
b_2 = -b_0, \quad b_3 = -\frac{1}{3} b_1
\]

and

\[
b_{j+2} = \frac{j(j - 1) - 2}{(j + 2)(j + 1)} b_j = \frac{j - 2}{j + 2} b_j, \quad j = 2, 3, \ldots.
\]

As in the last example, the even-numbered coefficients are multiples of \( b_0 \):

\[
b_2 = -b_0, \quad b_4 = \frac{1}{4} b_2 = 0, \quad b_6 = \frac{6}{6} b_2 = 0, \ldots, \quad b_{2m} = 0, \ldots,
\]
and the odd-numbered coefficients are multiples of $b_1:

\[ b_3 = -\frac{1}{3} b_1, \quad b_5 = \frac{1}{5} b_1, \quad b_7 = \frac{3}{7} b_5 = -\frac{1}{5 \cdot 7} b_1, \quad b_9 = \frac{5}{9} b_7 = -\frac{1}{7 \cdot 9} b_1, \ldots \]

\[ b_{2m-1} = \frac{2m-3}{2m+1} b_{2m-3} = -\frac{1}{(2m+1)(2m-1)} b_1, \ldots \]

Note that the general formula for $b_{2m}$ works from $m = 2$ on, whereas the formula for $b_{2m-1}$ works starting from $m = 0$.

Substituting these values back into (S) and separating the terms involving $b_0$ from those involving $b_1$, we obtain

\[ x(t) = b_0 (1 - t^2) + b_1 \sum_{m=0}^{\infty} \left[ \frac{-1}{(2m+1)(2m-1)} \right] t^{2m-1}. \]

This expression will be valid provided we stay away from the zeros, $\pm 1$, of the leading coefficient of (H)—that is, for $|t| < 1$.

So far we have managed to obtain explicit formulas describing all the coefficients of our solutions. Sometimes the recurrence relations are sufficiently complicated that we cannot find a general pattern. Nonetheless, we can use the recurrence relations to calculate explicitly any specified finite number of coefficients. Since this usually allows us to obtain good approximations to the values of our solutions at specified points (see Section 6.4), we content ourselves with finding the first few terms of the solution.

---

**Example 6.3.3**

Solve about $t = 0$:

\[ [D^2 - tD + t]x = 0. \]

We look for solutions in the form

\[ x(t) = \sum_{k=0}^{\infty} b_k t^k. \]

Substitution into (H) yields

\[ \sum_{k=0}^{\infty} k(k-1)b_k t^{k-2} - \sum_{k=0}^{\infty} kb_k t^k + \sum_{k=0}^{\infty} b_{k+1} t^{k+1} = 0. \]

We match up powers by substituting $j = k - 2$ (or $k = j + 2$) in the first series, $j = k$ in the second, and $j = k + 1$ (or $k = j - 1$) in the third:

\[ \sum_{j=0}^{\infty} (j + 2)(j + 1)b_{j+2} t^j - \sum_{j=1}^{\infty} j b_j t^j + \sum_{j=1}^{\infty} b_{j-1} t^j = 0. \]

We separate the $j = 0$ term from the first series and combine terms:

\[ 2b_2 + \sum_{j=1}^{\infty} [(j + 2)(j + 1)b_{j+2} - jb_j + b_{j-1}] t^j = 0. \]

This can happen only if

\[ h_2 = 0 \]

and

\[ b_{j+2} = \frac{j b_j - b_{j-1}}{(j + 2)(j + 1)}, \quad j = 1, 2, \ldots \]

We can use (R) to find the first few coefficients:

\[ b_3 = \frac{1}{6} (b_1 - b_0), \]

\[ b_4 = \frac{1}{12} (2b_2 - b_1) = \frac{1}{12} b_1, \]

\[ b_5 = \frac{1}{20} (3b_3 - b_2) = \frac{1}{40} (b_1 - b_0). \]

Thus, the first few terms of our solution are

\[ x(t) = b_0 + b_1 t + 2t^2 + \frac{1}{6} (b_1 - b_0) t^3 - \frac{1}{12} b_1 t^4 + \frac{1}{40} (b_1 - b_0) t^5 + \cdots \]

\[ = b_0 \left( 1 + \frac{1}{6} t^3 - \frac{1}{40} t^5 + \cdots \right) + b_1 \left( t + \frac{1}{6} t^3 - \frac{1}{12} t^4 + \frac{1}{40} t^5 + \cdots \right). \]

The expression is valid for all $t$. 

---
If \( t = 0 \) is a singular point, or if we want to match initial conditions at an ordinary point \( t_0 \neq 0 \), then we may wish to express our solutions as power series about \( t = t_0 \). Since most of us find it easier to work with a series about \( t = 0 \), the usual procedure is to first make the substitution \( T = t - t_0 \), as shown in the following example.

**Example 6.3.4**

Find the solution of

\[(D^2 - (t - 2))x = 0\]

subject to the condition at \( t = 2 \)

\[x(2) = 1, \quad x'(2) = 0.\]

Since the condition is to be satisfied at \( t = 2 \), we look for a series about \( t = 2 \):

\[(S) \quad x(t) = \sum_{k=0}^{\infty} b_k(t-2)^k.\]

Our condition tells us that

\[b_0 = 1, \quad b_1 = 0.\]

To simplify our calculations, we make the substitution

\[T = t - 2 \quad \text{(or) } t = T + 2,\]

which changes (H) and (S) to

\[(H') \quad [D^2 - T]x = 0\]

and

\[(S') \quad x(T+2) = \sum_{k=0}^{\infty} b_k T^k.\]

Substituting (S') into (H') we obtain

\[\sum_{k=2}^{\infty} k(k-1)b_k T^{k-2} - \sum_{k=3}^{\infty} b_k T^{k+1} = 0.\]

The substitutions \( j = k - 2 \) (or \( k = j + 2 \)) in the first series and \( j = k - 1 \) (or \( k = j + 1 \)) in the second, together with a separation of the \( j = 0 \) term in the first series, lead to

\[2b_2 + \sum_{j=1}^{\infty} [(j + 2)(j + 1)b_{j+2} - b_{j-1}]T^j = 0.\]

This implies

\[b_2 = 0\]

and

\[b_{j+2} = \frac{1}{(j + 2)(j + 1)} b_{j+1}, \quad j = 1, 2, 3, \ldots.\]

We know specific values for \( b_0, b_1, \) and \( b_2 \) and have a formula for \( b_{j+2} \) in terms of the coefficient three steps back, \( b_{j+1} \). Since \( b_1 \) and \( b_2 \) are both zero, we have

\[0 = b_4 = b_7 = \ldots = b_{3m+1} \ldots.\]

\[0 = b_5 = b_8 = \ldots = b_{3m-2} \ldots.\]

The first few terms of the form \( b_{3m} \) are

\[
\begin{align*}
b_6 &= 1, & b_9 &= \frac{1}{3 \cdot 2}, & b_{6} &= \frac{1}{3 \cdot 2}, & b_{6} &= \frac{1}{6 \cdot 5}, & b_{6} &= \frac{1}{6 \cdot 5 \cdot 3 \cdot 2}, \\
b_9 &= \frac{1}{9 \cdot 8}, & b_{9} &= \frac{1}{9 \cdot 8}, & b_{6} &= \frac{1}{6 \cdot 5 \cdot 3 \cdot 2}.
\end{align*}
\]

The denominator of \( b_{3m} \) looks like \((3m)!\), except that it is missing the factors \( 1, 4, 7, \ldots, 3m - 2 \). Thus

\[b_{3m} = \frac{1 \cdot 4 \cdot 7 \cdots (3m - 2)}{(3m)!}.
\]

This general formula starts to work at \( m = 1 \).

Now putting our expressions back into (S'), we get

\[x(T+2) = 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3m - 2)}{(3m)!} T^{3m}
\]
or using $T = t - 2$,

$$x(t) = 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3m - 2)}{(3m)!} (t - 2)^{3m}.$$ 

The expression is valid for all $t$.

This technique also works nicely for nonhomogeneous equations, as in our final example.

Example 6.3.5

Solve about $t = 0$

(N) $$[D^3 + tD + 1]x = t.$$ 

We seek solutions in the form

(S) $$x(t) = \sum_{k=0}^{\infty} b_k t^k.$$ 

Substituting (S) into (N) yields:

$$\sum_{k=2}^{\infty} k(k - 1)b_k t^{k-2} + \sum_{k=1}^{\infty} kb_k t^k + \sum_{k=0}^{\infty} b_k t^k = t.$$ 

We bring the forcing term to the left-hand side, make index substitutions ($j = k - 2$, $j = k$, $j = k$) to match up powers of $t$, and separate the $j = 0$ and $j = 1$ terms to obtain

$$(2b_2 + b_0) + (6b_3 + 2b_1 - 1)t + \sum_{j=2}^{\infty} [(j + 2)(j + 1)b_{j+2} + (j + 1)b_j] t^j = 0.$$ 

This leads to

$$b_2 = -\frac{1}{2} b_0, \quad b_3 = \frac{1}{5} - \frac{1}{3} b_1,$$

and

$$b_{j+2} = -\frac{1}{j + 2} b_j, \quad j = 2, 3, \ldots$$

We use (R) to express the even-numbered coefficients in terms of $b_0$ and the odd-numbered ones in terms of $b_1$:

$$b_2 = -\frac{1}{2} b_0, \quad b_4 = -\frac{1}{4} b_2 = -\frac{1}{4} \cdot \frac{1}{2} b_0, \quad b_6 = -\frac{1}{6} b_4 = -\frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} b_0, \ldots$$

$$b_{2m} = \frac{(-1)^m}{(2m)!} \cdot 6 \cdot 4 \cdot 2 \cdot b_0, \ldots$$

$$b_3 = -\frac{1}{3} b_1 + \frac{1}{6}, \quad b_5 = -\frac{1}{5} b_3 = -\frac{1}{5} \cdot \frac{1}{3} b_1 - \frac{1}{5} \cdot \frac{1}{3},$$

$$b_7 = -\frac{1}{7} b_5 = -\frac{1}{7} \cdot \frac{1}{5} \cdot \frac{1}{3} b_1 + \frac{1}{7} \cdot \frac{1}{5} \cdot \frac{1}{3}, \ldots$$

$$b_{2m+1} = \frac{(-1)^m}{(2m + 1)!} \cdot 6 \cdot 4 \cdot 2 \cdot b_1 + \frac{(-1)^{m+1}}{(2m + 1)!} \cdot 6 \cdot 4 \cdot 2 \cdot b_0, \ldots$$

Substitution in (S) gives us

$$x(t) = b_0 \left( 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} \cdot 6 \cdot 4 \cdot 2 \cdot 2^{2m} \right) t^{2m} + b_1 \left( t + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m + 1)!} \cdot 6 \cdot 4 \cdot 2 \cdot 2^{2m} \right) t^{2m+1} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m + 1)!} \cdot 6 \cdot 4 \cdot 2 \cdot 2^{2m+1}.$$ 

The expression is valid for all $t$.

Although all our examples were second-order o.d.e.'s, the same procedure works about ordinary points for o.d.e.'s of any order.

**SOLUTIONS ABOUT ORDINARY POINTS**

Suppose the coefficients $a_i(t)$ and forcing term $E(t)$ in the o.d.e.

(N) $$[a_0(t)D^n + \cdots + a_0(t)] x = E(t)$$

are all polynomials. We call $t = t_0$ an ordinary point of (N) if $a_0(t_0) \neq 0$ and a singular point if $a_0(t_0) = 0$.

If $t = 0$ is an ordinary point of (N), we can find a series expression

(S) $$x(t) = \sum_{k=0}^{\infty} b_k t^k$$

for the solutions of (N) as follows:
1. Substitute (S) into (N).
2. Combine terms so as to rewrite the equation in the form of a power series set identically equal to zero. (This requires index substitutions to match up powers of \( t \), and possibly separating out a few extra terms to match up the limits of summation.)
3. Set each (combined) coefficient of the resulting series equal to zero, to obtain a recurrence relation (R) between the general coefficient in (S) and earlier ones.
4. If a pattern is evident, write down an explicit formula for the coefficients of (S); otherwise, we can still use (R) to write down any specified finite number of coefficients.
5. Substitute the coefficients back into (S) to obtain a power series expression for the solutions.

The resulting series expression will be valid for \(| t | < R \), provided the leading coefficient \( a_{0}(t) \) of (N) has no (complex) zeros with \(| z | < R \).

If \( t = t_0 \) is a nonzero ordinary point of (N), we can find a series expression about \( t = t_0 \)

\[
(S) \quad x(t) = \sum_{k=0}^{\infty} b_k(t-t_0)^k
\]

for the solutions of (N). We substitute \( T = t - t_0 \) (or \( T = t + t_0 \)) and use the procedure described in steps 1 through 5 to find a solution

\[
(S') \quad x(T + t_0) = \sum_{k=0}^{\infty} b_k T^k
\]

to the modified o.d.e.

\[
(N') \quad [a_0(T + t_0)D^n + \cdots + a_n(T + t_0)]x = E(T + t_0).
\]

We then substitute \( T = t - t_0 \) back into (S') to get (S).

Note

Concerning our hypotheses

The assumption in this section that the coefficients and forcing term in (N) are polynomials can be weakened. The general theorem, of which we quoted a special case, is the following.

Theorem: Suppose that

1. \( a_0(t), \ldots, a_n(t) \), and \( E(t) \) are analytic functions whose power series expressions about \( t = t_0 \) are all valid (at least) for \(| t - t_0 | < R \), where \( R > 0 \); and

2. \( a_n(z) \neq 0 \) for any complex number \( z \) satisfying \(| z - t_0 | < R \).

Then every solution of \( a_0(t)D^n + \cdots + a_n(t)x = E(t) \) is analytic at \( t = t_0 \), with a power series expression about \( t = t_0 \) valid (at least) for \(| t - t_0 | < R \).

The procedure described in the summary actually works under these more general hypotheses. We replace each \( a_i(t) \) and \( E(t) \) with its power series expression and then proceed as before. However, if one (or more) of the coefficients is not a polynomial, then step 1 will require multiplication of series (see Note 1 in Section 6.2 and Exercises 18 and 19). Also, we cannot in general expect a simple recurrence relation for the \( b_k \)'s; we can hope to actually calculate only the first few terms.

EXERCISES

In Exercises 1 through 6, (a) find the recurrence relation for the coefficients of series solutions about \( t = 0 \), and (b) write out the terms to \( r^2 \) of the general solution.

1. \( [D^2 + (1 + 4)D + 4]x = 0 \) 2. \( [D^2 + 4]x = 0 \)

3. \( [D^2 + 2D + 2]x = 0 \) 4. \( [D^2 + 2D + 1]x = 0 \)

5. \( [D^2 + 3D - 4]x = 0 \) 6. \( [D^2 + 4]x = 0 \)

In Exercises 7 through 10, (a) find the recurrence relation for the coefficients of series solutions about \( t = 0 \), and (b) write out the terms to \( r^4 \) of the solution matching the given initial condition. Compare the cited exercise in Section 6.1.

7. \( [D^2 + (10 + 1)D + 10]x = 10; \quad x(0) = x'(0) = 0 \) (Exercise 1)

8. \( [10D^2 + 10D + (10 - 1)D + 10]x = 0; \quad x(0) = 1, x'(0) = 0 \) (Exercise 2)

9. \( [10D^2 + (10 - 1)D + 10]x = 0; \quad x(0) = 1, x'(0) = 0 \) (Exercise 3)

10. \( [10 - 1)D^2 + 10D + 10]x = 0; \quad x(0) = 1, x'(0) = 0 \) (Exercise 4)

In Exercises 11 and 12, (a) find the recurrence relation for the coefficients of series solutions about \( t = t_0 \), and (b) write out the terms to \( (t - t_0)^4 \) of the general solution.

11. \( [D^2 + 4]x = 0; \quad t_0 = 1 \) 12. \( [D^2 + 4]x = 0; \quad t_0 = 2 \)

In Exercises 13 and 14, (a) find the recurrence relation for the coefficients of series solutions about \( t = 0 \), and (b) find the general form for the coefficients.

13. \( [D^2 + (D + 1)x = 0 \) 14. \( [(t^2 + 1)D^2 - 2]x = 0 \)

Exercises 15 through 17 deal with some important equations from mathematical physics.

15. a. Find the recurrence relation for solutions about \( t = 0 \) of the Legendre equation \( \left[(r^2 - 1)D^2 + 2(r - 1)D + \mu(r + 1)]x = 0, \right \) where \( \mu \) is a constant.

b. Show that if \( \mu \) is a nonnegative integer, then this o.d.e. has a polynomial solution of degree \( \mu \).

16. a. Find the recurrence relation for solutions of the Hermite equation \( [D^2 - 2D + \mu x = 0, \) where \( \mu \) is a constant.

b. Show that if \( \mu = 2N \) where \( N \) is a nonnegative integer, then this o.d.e. has a polynomial solution of degree \( N \).