Math 218 — Differential Forms in Algebraic Topology — Spring 2009

Course Information

Block: F+TR, Tue Thu 12:00-1:15 p.m.
Instructor: Loring Tu
Email: loring.tu@tufts.edu
Office: Bromfield-Pearson 206
Office hours (Fall 2008): Mon 10:30–11:30 a.m., Tue, Thu 1:30–2:20 p.m.
Phone: 7-3262

Prerequisites: Math 217, or the first twenty-two chapters of the book *An Introduction to Manifolds*, or consent

Text: Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, 3rd corrected printing, Springer, New York, 1995.

Course description:
The central problem in topology is to decide if two spaces are homeomorphic or if two manifolds are diffeomorphic. Algebraic topology offers a possible solution by transforming the geometric problem into an algebraic problem. To accomplish this, one associates to each topological space an algebraic object such as its cohomology vector space so that homeomorphic topological spaces correspond to isomorphic vector spaces. Such an association is called a *functor*. The cohomology functor greatly simplifies the original problem, for the dimension alone determines the isomorphism class of a finite-dimensional vector space. This gives a simple necessary condition for two topological spaces to be homeomorphic.

Over the past one hundred years our forefathers have discovered many useful functors, among which are the fundamental group, higher homotopy groups, homology groups, and cohomology rings. There are various versions of cohomology—singular, de Rham, and Čech, among others. This leads naturally to two new problems: (1) How does one compute these functors? (2) What are the relations that exist among the different functors? Algebraic topology, as it is usually practiced, concerns itself with these questions in the continuous category, with topological spaces and continuous maps. Its great generality makes it applicable in a variety of settings, but also renders many of its constructions difficult and inaccessible to the novice.

The guiding principle in this course is to use differential forms as an aid in exploring some of the less digestible aspects of algebraic topology. We will work primarily but not exclusively with smooth manifolds. This has the advantage of simplifying the theory as well as the constructions; for example, all three versions of cohomology coincide on smooth manifolds. It allows us to study topics that are normally relegated to a second course in algebraic topology, such as Poincaré duality, the Thom isomorphism, and spectral sequences. These topics will respond to the two problems mentioned above: (1) techniques of computation, (2) theorems relating the various functors.

More specifically, the topics to be covered are the following: de Rham cohomology, cohomology with compact support, Mayer–Vietoris sequence, Poincaré lemma on the cohomology of $\mathbb{R}^n$, degree of a proper map, Brouwer fixed-point theorem, hairy-ball theorem, Poincaré duality for a manifold, Künneth formula for the cohomology of a product, Thom isomorphism for the cohomology of a vector bundle, Euler class, the nonorientable case, Čech cohomology, isomorphism between de Rham and Čech cohomology, presheaves, Hopf index theorem on the zeros of a vector field, spectral sequences, cohomology with integer coefficients, de Rham's theorem, and applications to homotopy theory.