NONALGEBRAIC CONTACT ANOSOV FLOWS ON 3-MANIFOLDS

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ABSTRACT. Geodesic flows of Riemannian or Finsler manifolds have been the only known contact Anosov flows. We show that even in dimension 3 the world of contact Anosov flow is vastly larger via a surgery construction near a transverse Legendrian knot that encompasses both the Handel–Thurston and Goodman surgeries and produces flows that are not topologically orbit equivalent to any algebraic flow. This likely includes examples on many hyperbolic 3-manifolds, any of which have a remarkable array of dynamical and geometric properties.

CONTENTS

1. INTRODUCTION

1.1. **Algebraic and anomalous Anosov flows.** Geodesic flows of Riemannian (or Finsler) manifolds of negative curvature have been a classical and central proving ground in hyperbolic dynamics. That they preserve a natural smooth measure is one of the consequences of their Hamiltonian nature, but it can also be seen as a consequence of the fact that they preserve a natural contact form $p\,dq$, a smooth flow-invariant form A such that $A \wedge dA \wedge \cdots \wedge dA$ defines a (clearly smooth invariant) volume.

REMARK 1.1. While every Anosov flow admits a canonical invariant 1-form (defined by being 1 on the generating vector field and having its kernel spanned by the strong stable and unstable subspaces), this is only rarely smooth [37, Theorem 4.7], [18], and for suspensions, this canonical 1-form is smooth but closed,

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so $A \wedge dA \wedge \cdots \wedge dA = 0$; the extreme opposite of the contact case. Thus, the contact condition can be viewed as a constraint on the canonical 1-form.

For the discrete-time counterpart, Anosov diffeomorphisms, all known examples are topologically equivalent to a linear representative, and thereby the known situations are classified by a list of algebraic objects. Anosov flows are not this rigid [19, 11, 24, 22]. However, some experts thought that surgeries could not produce *contact* flows, and the published record (see, e.g., [23, 17, 31, 12]) has a paucity of examples. We remark that Anosov flows on the unit tangent bundle of a surface are always topologically orbit equivalent to an algebraic flow [20]. The present work (announced in [17]) is the first to present genuinely new contact Anosov flows, and it produces a wealth of them, showing that the universe of contact flows is vastly greater than that of geodesic flows of Riemannian or Finsler manifolds.

That contact Anosov flows need not be topologically orbit equivalent to an algebraic flow means there are no "shortcuts" in study of their structure. For instance, the proof that smoothness of the invariant foliations implies smooth, hence in particular topological, conjugacy to a standard algebraic system, up to a canonical time-change [1, 2] had to laboriously exclude the Lie-algebraic structures that do not fit into that collection of classical examples. It would have been greatly simplified by knowing a priori that contact flows are topologically orbit equivalent to an algebraic flow.

THEOREM 1.2. *There are contact Anosov flows that are not topologically orbit equivalent to algebraic flows.*

1.2. **The main result.** While Theorem 1.2 is our essential conclusion, there is great additional value both in the remarkable properties of the examples obtained when starting with a Legendrian knot having hyperbolic complement and in the method: a surgery that produces many new examples of volume-preserving Anosov flows on 3-manifolds and, upon a time-change, new examples of contact Anosov flows, and which subsumes both the Handel–Thurston [24] and Goodman [22] surgeries, the latter of which was invented to deal with suspensions of toral automorphisms.

THEOREM 1.3. *For a contact Anosov flow on a 3-manifold M and a transverse Legendrian knot (Definition 2.1) in M, there is a family of smooth Dehn surgeries that produce new contact Anosov flows.*

If the original contact flow is the geodesic flow on the unit tangent bundle of a negatively curved surface, then these surgeries include the Handel–Thurston surgery, in which case the resulting flow is not topologically orbit equivalent to an algebraic flow. In particular:

1. *The flow acts on a manifold that is not a unit tangent bundle.*
2. *Its Anosov splitting (into stable and unstable subbundles) is not $C^{1+zygmund}$, i.e., does not have "little Zygmund" (hence not Lipschitz-continuous) derivative.*
3. *Its topological and volume entropies differ.*

Here, item 2. uses

THEOREM 1.4 ([29, 21])**.** *If a contact Anosov flow on a 3-manifold has* $C^{1+zygmund}$ *Anosov splitting, then it is smoothly conjugate to a geodesic flow.*

3. follows from

THEOREM 1.5 ([17])**.** *Contact Anosov flows on 3-manifolds for which the topological and volume entropies coincide are up to finite covers smoothly orbit equivalent to a geodesic flow.*

We now describe how one might obtain contact Anosov flows on hyperbolic manifolds and what this implies.

THEOREM 1.6. *If complement in* M *of the transverse Legendrian knot in Theorem 1.3 is a hyperbolic manifold, then all but finitely many of our Dehn surgeries produce a hyperbolic manifold (and hence neither the unit tangent bundle of any surface nor a torus bundle over a circle), and any contact Anosov flow that is topologically orbit equivalent to the resulting contact Anosov flow has the following properties.*

1. *It is not topologically orbit equivalent to an algebraic flow.*
2. *It is not quasigeodesic.*
3. *Its orbits are geodesics for suitable Riemannian metrics on* M*.*
4. *Each of its closed orbits is freely homotopic to infinitely many others* [13, Theorem A] *(while for algebraic flows each homotopy class only has one closed orbit). The partition of such a homotopy class by isotopy classes is nontrivial (i.e., has more than one element and does not consist of singletons) [Fenley, personal communication].*
5. *It is associated with a new example of a quasigeodesic pseudo-Anosov flow (see* [15], [43, Section 5]*).*

Here item 5. refers to the following notion.

DEFINITION 1.7. A *quasigeodesic* curve is one that is efficient, up to a bounded multiplicative distortion, in measuring distances in relative homotopy classes, and a flow is said to be quasigeodesic if all flow lines are quasigeodesics [14].

REMARK 1.8. Although we do not provide any situations in which the transverse Legendrian knot can be chosen to have a hyperbolic complement, it seems likely that this can be done. For now it remains a problem to prove it.

PROBLEM. Give an example of a contact 3-manifold containing a Legendrian knot with hyperbolic complement.

One promising directions seems to be as follows.

PROBLEM. Show that for the geodesic flow of a negatively curved surface, a vector field perpendicular to a geodesic that fills the surface constitutes such a knot.

This would entirely suffice for purposes of finding a wealth of contact Anosov flows on hyperbolic manifolds, but even greater wealth could arise from a sufficiently broad solution of the following.

PROBLEM. Give conditions on a contact 3-manifold for the existence of a Legendrian knot with hyperbolic complement.

REMARK 1.9. The Handel–Thurston examples are not *topologically orbit equivalent* to a standard model. As the introduction to [24] explains:

> The standard examples of Anosov flows on three-dimensional manifolds are the geodesic flows on surfaces of negative curvature, and the suspensions of Anosov diffeomorphisms of T^2.
>
> Both of these families are algebraic; i.e., they fit into the broader category of one-parameter subgroups acting on homogeneous spaces. More precisely, a flow φ_t on a compact connected manifold M is algebraic if $M = \Gamma\backslash G/K$, where G is a Lie group, K is a compact subgroup, and Γ is a discrete cocompact subgroup of G acting by left multiplication; the flow is given by φ_t: $\Gamma g K \mapsto \Gamma g(\exp t\alpha) K$ where α is an element of the Lie algebra of G.
>
> Tomter showed that if M is three dimensional and if φ_t is an algebraic Anosov flow, then some finite cover of M is either a geodesic flow or the suspension of a toral diffeomorphism [T] (Tomter called these flows (G, Γ)-induced). By examining the fundamental groups of the manifolds we construct, we show that our examples are not algebraic. These are the first such examples.

Since Handel and Thurston obtain their conclusion by examining the fundamental group, their examples are not topologically orbit equivalent to any algebraic flow. Likewise, those of our examples whose phase space is a hyperbolic manifold are not algebraic in this sense. Indeed, Handel and Thurston remark that "it would be very interesting to find examples of Anosov flows... on an atoroidal M". (For the present context it suffices to know that "hyperbolic" implies "atoroidal".) This was accomplished in [22], and we show how do so in the contact category, provided one takes a suitable Legendrian knot as a starting point.

The first author has previously presented this construction in lectures, and this is reflected in the literature (e.g., [4, page 53], [5, page 754], [12, page 1191], [25, page 254], [34, page 1417]).

The construction presented here is of interest from entirely different points of view. While the motivation is dynamical, it can be viewed as surgery that produces particular kinds of contact structures on new manifolds. It may be an interesting project to study these contact structures. A separate aspect of interest is that our surgery is carried out along a Legendrian knot, not in particularly close proximity to a periodic orbit (see also [9]). Our context is one that makes many such knots available, and, on the other hand, for any one such knot we describe entire families of surgeries.

1.3. **Methods.** Both Handel–Thurston and Goodman were inspired by the idea of Dehn surgery. It is of particular interest to point to [22, p. 307], which describes something along rather similar lines to what we do here, but starting from a suspension (of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$) and hence in a context that differs from ours in a fundamental way—in particular, the orbit of the origin gives the knot with hyperbolic complement. We recall that the Handel–Thurston examples are graph manifolds, i.e., they are obtained by glueing Seifert fiber spaces together along their $\mathbb{T}^2$ boundary components.

The method for showing that the surgery proposed here yields contact flows is new: It involves a suitable deformation of the contact form and a time-change that is finely adapted to this deformation. Showing that the surgered and time-changed flows are hyperbolic was essentially done in [24] and [22], but we outline a proof by Thierry Barbot that is remarkably clean, direct and transparent [4, page 54].

We should also mention that the construction of contact *structures* is not our claim to novelty; there are contact structures on any 3-manifold [32]. What is new is that on a variety of manifolds we have a contact *form* that is invariant under a nonstandard Anosov flow.

1.4. **Structure of this paper.** Below we prove Theorem 1.6. Section 2 describes the local structure near the knot used to define the surgery, and Section 3 introduces our contact surgery. Section 4 shows that this yields contact Anosov flows. Section 5 points out that our surgery subsumes and extends the surgeries of Handel–Thurston and Goodman; it serves to convey that this surgery provides an abundance of examples. Section 6 recalls that in the Handel–Thurston situation we get nonalgebraic flows. An appendix translates the Alexeev cone criterion for hyperbolic into the language of Lyapunov–Lorentz metrics.

1.5. **Contact Anosov flows on hyperbolic manifolds.** In this section we prove Theorem 1.6.

Although we describe it in terms of an annulus, our surgery can also be viewed as the deletion of a knot (or tubular neighborhood thereof) followed by what is called a *Dehn filling*, that is, the glueing-in of a solid torus subject to a prescribed identification of a meridian on the boundary of the knot complement with a curve on the boundary of the inserted torus. The effect of this surgery is described uniquely by the slope of the boundary curve. Therefore, these surgeries are parameterized by $\mathbb{Q}\cup\{\frac{1}{0}\}$. If the initial knot can be chosen so as to have a hyperbolic complement, then the abundance of hyperbolic manifolds arising from our construction is due to the following result of Thurston's.

THEOREM 1.10 ([41, Theorem 5.8.2],[42, 36])**.** *For all but finitely many slopes, Dehn filling a hyperbolic 3-manifold gives rise to a hyperbolic manifold.*

Some orbits of contact Anosov flows on hyperbolic manifolds are quite far from being globally minimizing (Theorem 1.3.2). This follows from

THEOREM 1.11 ([13, Theorem B], [14, Theorem A])**.** *Contact Anosov flows on hyperbolic manifolds are not quasigeodesic.*

This property is in contrast with item 3. in Theorem 1.6 that our contact flows are *geodesible*. This does not use hyperbolicity; we stated it in Theorem 1.6 to achieve this juxtaposition.

PROPOSITION 1.12. *If X generates a flow that preserves a contact form A, and g is a Riemannian metric for which $X \perp \ker A$ and $g(X,X) \equiv 1$, then the orbits are geodesics for g.*

REMARK 1.13. This holds in the hyperbolic case, which is not quasigeodesic. Thus, the metrics in Proposition 1.12 are not negatively curved in this case.

Proof. To verify the geodesic equation $\nabla_X X = 0$, where ∇ is the Levi-Cività derivative, we show that $g(\nabla_X X, \xi) = 0$ for any vector field ξ. We compute the Lie derivative

$$\mathscr{L}_X g(X,\xi) = \nabla_X g(X,\xi) = g(\nabla_X X,\xi) + g(X,\nabla_X \xi).$$

Since $g(X,\xi) = A(\xi)$, this agrees with

$$\mathscr{L}_X A(\xi) = A([X,\xi]) = A(\nabla_X \xi - \nabla_\xi X) = g(X,\nabla_X \xi) - g(X,\nabla_\xi X).$$

Therefore,

$$g(\nabla_X X,\xi) = -g(X,\nabla_\xi X) = -\frac{1}{2}\nabla_X g(X,X) = -\frac{1}{2}\nabla_X 1 = 0. \qquad \square$$

Finally, a contact flow is $\mathbb{R}$-covered [6, Theorem A] and on an atoroidal manifold has a transverse regulating pseudo-Anosov flow [10, Corollary 5.3.16], which is quasigeodesic [16, Theorem 6.1, §7]. This proves item 5. of Theorem 1.6.

2. LOCAL STRUCTURE NEAR LEGENDRIAN KNOTS

DEFINITION 2.1. A *Legendrian curve* in a contact manifold is a curve tangent to the kernel of the contact form at every point. In the presence of a contact Anosov flow, a Legendrian curve (which is by construction transverse to the flow) is said to be *transverse* if it is also transverse to both the strong stable and strong unstable subbundles E^- and E^+ of the flow.

PROPOSITION 2.2. *Suppose φ is a contact Anosov flow with orientable strong-stable subbundle E^- and γ is a transverse Legendrian knot.*

Then there is a smooth annular transversal Σ through γ that is transverse to E^- and E^+ away from γ, and there are coordinates (s,w) on Σ such that $s \in S^1$ is the parameter for γ, and on $\Lambda := \bigcup_{t\in(-\eta,\eta)} \varphi^t(\Sigma)$ the contact form A satisfies

$$A = dt + w\,ds, \quad dA = dw \wedge ds \quad \text{and} \quad A \wedge dA = dt \wedge dw \wedge ds.$$

Here, t denotes the transverse parameter given by the flow.

REMARK 2.3. This in particular implies that Σ is transverse to the flow, which means that our surgery does indeed reconnect orbits—without transversality it would not even be clear whether orbits become smooth curves after surgery.

Proof. Start with the Legendrian knot $\gamma\colon S^1 \to M$, $s \mapsto \gamma(s)$ and $\epsilon > 0$. Consider the transversals

$$T^- := \bigcup_{s\in S^1} W^-_\epsilon(\gamma(s)) \text{ and } T^+ := \bigcup_{s\in S^1} W^+_\epsilon(\gamma(s))$$

spanned by local stable and unstable manifolds. Note that $\ker A$ is the common tangent plane of T^+ and T^- at points of γ and that T^+ and T^- are fibered by smooth Legendrian curves (stable and unstable leaves) transverse to γ. Using,

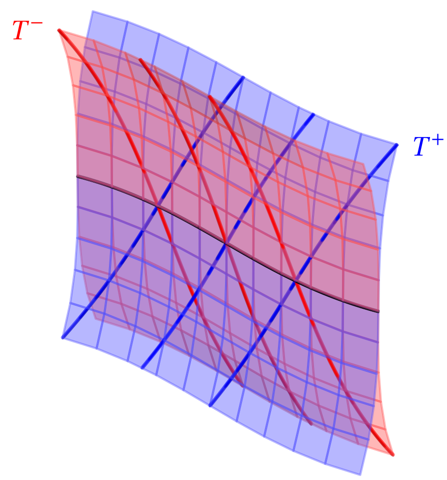

FIGURE 1. Transversals

for instance, local exponential maps centered at $\gamma(s)$ and a partition of unity we obtain a smooth convex combination Σ of these transversals that contains γ such that

- $T_{\gamma(s)}\Sigma = \ker A$ for all $s \in S^1$,
- Σ is transverse to both E^- and E^+ off γ,
- Σ is smoothly fibered by Legendrian curves transverse to γ.

Next, we define convenient coordinates on Σ. One coordinate on Σ is the parameter $s \in S^1$ of the Legendrian knot γ, another is a smoothly chosen parameter v along the Legendrian curves transverse to γ. In these coordinates we have

$$A = dt + f(s,v)\,ds.$$

Since γ is a Legendrian curve we have $A_{(0,s,0)}(\partial/\partial s) = 0$, i.e., $f(s,0) = 0$.

Moreover,

$$0 \neq dA = (\partial/\partial v) f(s,v)\,dv \wedge ds,$$

so $(\partial/\partial v) f(s,v) \neq 0$. Thus, the transformation

$$(s,v) \mapsto (s, f(s,v)) =: (s,w)$$

is nonsingular and gives coordinates on Σ that are as desired. □

3. CONTACT SURGERY

The surgeries we describe in this work are topologically of the *Dehn filling* type: One can view them as removing a tubular neighborhood of a knot and glueing that solid torus back in in such a way as to match a prescribed closed curve on the boundary with a meridian. The surgery is then described by the *slope* of the prescribed curve, a pair of coprime integers. For a pair $(1,n)$ this can equivalently be described as splitting apart an annulus in the manifold and glueing both copies of this annulus back together with a shear, i.e., by again identifying the 2 annuli via the shear map. We remark that although this shear map is a homeomorphism of the annulus, this defines a discontinuous operation since the resulting space is no longer homeomorphic to the original one. We perform surgery of the latter type, but have occasion to keep in mind the knot near which the surgery takes place.

Moreover, it is important for us that the surgery yield a smooth manifold, not just a topological one, and moreover, that the vector field that generates the flow gives rise to a well-defined vector field on the surgered manifold.

Proposition 2.2 gives an annulus that *contains* the Legendrian knot γ and is uniformly transverse to the flow. (By moving γ along the flow one obtains a homotopic knot γ' with a tubular neighborhood whose boundary contains this annulus; this tubular-knot-neighborhood point of view will prevail in Section 6, where we study topological properties.) We split the flow-box chart from Proposition 2.2 into 2 one-sided flow-box neighborhoods of the surgery annulus, and while the initial transition map between these on $\{0\}\times S^1\times(-\epsilon,+\epsilon)$ is the identity, the surgered manifold is defined by imposing the desired shear as the transition map on this annulus. The use of flow-box charts ensures that the original vector field defining the contact Anosov flow defines a smooth vector field on the surgered manifold, i.e., that the orbits are reglued to smooth curves.

The transition map pulls meridians around the equator $q\in\mathbb{N}$ times before exiting (Figure 2).

DEFINITION 3.1. The contact $(1,q)$-Dehn surgery for $q\in\mathbb{N}$ is defined by imposing on the aforementioned chart overlap the transition map

$$F\colon S^1\times(-\epsilon,\epsilon)\to S^1\times(-\epsilon,\epsilon),\quad (s,w)\mapsto(s+f(w),w). \tag{1}$$

Here, (Figure 6)

$$f\colon[-\epsilon,\epsilon]\to S^1,\quad w\mapsto qg(w/\epsilon),$$

where $g\colon\mathbb{R}\to[0,2\pi]$ is a monotone smooth function with $0\le g'\le 4$ such that $g((-\infty,-1])=\{0\}$ and $g([1,\infty))=\{2\pi\}$ (Figure 6).

Because of our use of flow-box charts it is apparent that the vector field generating the original contact Anosov flow defines a smooth vector field X on the surgered manifold M. To see whether the original contact form induces a contact form on the surgered manifold we need to check whether the coordinate representation $A=dt+w\,ds$ is preserved by the transition map F between the two overlapping charts that define the surgery. Unfortunately, this is not so:

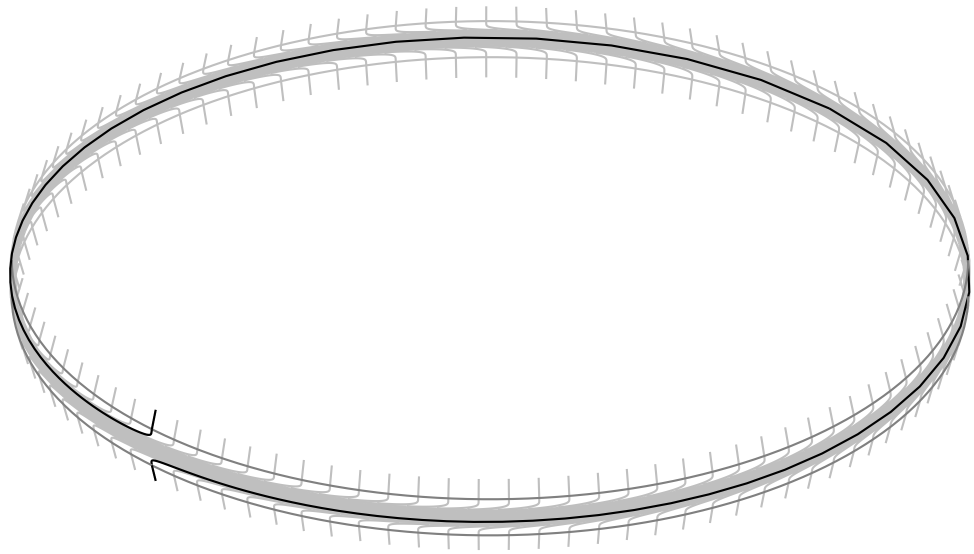

FIGURE 2. Surgery in the chart (with $q = 1$)

LEMMA 3.2. *$F_* A = A + w f'(w)\,dw$ and $F_* dA = dA$.*

Proof. Since $d(s + f(w)) = ds + f'(w)\,dw$ we have

$$F_* A = dt + w\,d(s + f(w)) = A + w\,f'(w)\,dw$$

and

$$F_* dA = dw \wedge d(s + f(\theta)) = dw \wedge ds = dA. \qquad \square$$

This implies that dA induces a well-defined (albeit not necessarily exact) 2-form on the surgered manifold. Moreover, since $F_* dt = dt$, we see that $A \wedge dA = dt \wedge dA$ is well-defined on the surgered manifold. Indeed, we have

COROLLARY 3.3. *The new flow preserves the Liouville volume defined by $A \wedge dA$.*

We remark that Handel and Thurston also obtained volume-preserving flows.

4. CONTACT ANOSOV PROPERTY

We produce a contact form on the surgered manifold by giving representations A_- and A_+ for it on the 2 one-sided flow-box charts (whose overlap defines the surgery annulus) that on one hand agree with $A = dt + w\,ds$ near the chart boundary and on the other hand are such that $F_* A_- = A_+$. This follows from the existence of a function h such that $F_*(A - dh) = A + dh$: deforming A to $A - dh$ on one side of the surgery and to $A + dh$ on the other gives a contact form that glues together nicely, and dA is unchanged. To define h recall that $F_* A = A + w f'(w)\,dw$ and “split the difference”: To get $dh = {}^1/_2 w f'(w)\,dw$ at points where $F \neq \mathrm{Id}$ set

$$h(t,w) := \frac{1}{2}\lambda(t)\int_0^w x f'(x)\,dx \tag{2}$$

on $(-\eta,\eta)\times S^1$ and $h=0$ outside. Here, $\lambda\colon \mathbb{R}\to[0,1]$ is a smooth bump function

- supported in $(-\eta,\eta)$,
- with $\lambda^{-1}(\{1\})$ an interval containing a neighborhood of 0,
- monotone on the two intervals where it takes intermediate values,
- with $|\lambda'| \le \pi/\eta$.

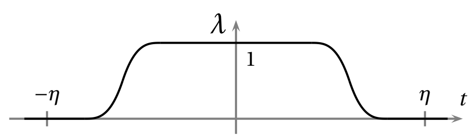

FIGURE 3. The function λ

(2) defines a smooth function on M since $h=0$ for t near $\pm\eta$ by choice of λ.

LEMMA 4.1. *If we choose $0<\epsilon<\frac{\eta}{2\pi q}$, then $|dh(X)|<1$.*

Proof. Clear where $dh=0$. Elsewhere use $|f'(w)| = q|\frac{d}{dw}g(\frac{w}{\epsilon})| = \frac{q}{\epsilon}|g'(\frac{w}{\epsilon})|$, $|\lambda'|\le\pi/\eta$, $0\le g'\le 4$ to get

$$\left|\frac{\partial h}{\partial t}\right| = \left|\frac{1}{2}\lambda'(t)\int_0^w x f'(x)\,dx\right| \le \left|\frac{q\pi}{2\eta}\right|\left|\int_0^\epsilon x\frac{|g'|}{\epsilon}\,dx \le \left|\frac{q\pi}{2\eta}\epsilon\frac{4}{\epsilon}\int_0^\epsilon dx\right| = \frac{2q\pi}{\eta}\epsilon < 1. \quad \square$$

THEOREM 4.2. *The Dehn surgery of Definition 3.1 near a Legendrian knot and a choice of bump function as in (2) yield a contact flow φ_h^t defined by a vector field $X_h := \dfrac{X}{1\pm dh(X)}$, where the contact form is $A_h := A\pm dh$. Here, for both X_h and A_h, we take "+" on one chart and "−" on the other.*

Proof. That A_h is smooth is clear away from the surgery. At the surgery it follows once we check that $F_*(A-dh) = A+dh$. To that end observe first that $F_*dh = dh$ by (1) because $dh = \frac{w}{2}f'(w)\,dw$ on the surgery annulus. Then Lemma 3.2 yields

$$F_*(A-dh) = F_*A - F_*dh = (A+2dh) - F_*dh = A+dh.$$

Next, A_h is clearly a contact form where $dh=0$. Elsewhere,

$$\begin{aligned} A_h\wedge dA_h = (A\pm dh)\wedge d(A\pm dh) &= (A\pm dh)\wedge dA \\ &= (A\pm(\frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial w}dw))\wedge dA = (1\pm\frac{\partial h}{\partial t})dt\wedge dA \end{aligned}$$

by Lemma 5.1, and $1\pm\dfrac{\partial h}{\partial t}\neq 0$ by Lemma 4.1.

X_h is a smooth vector field since by Lemma 4.1 the denominator is nonzero, and "±" matches smoothly on the surgery annulus, near which $dh(X) = \partial h/\partial t = 0$.

Finally, the flow φ_h^t generated by X_h preserves A_h since

$$A_h(X_h) = A_h\Big(\frac{X}{1 \pm dh(X)}\Big) = \frac{(A \pm dh)(X)}{1 \pm dh(X)} = \frac{1 \pm dh(X)}{1 \pm dh(X)} = 1$$

implies $\mathscr{L}_{X_h} A_h = 0$. □

THEOREM 4.3. *The Dehn surgery of Definition 3.1 near a Legendrian knot with the choices of Theorem 4.2 yields a contact Anosov flow $\varphi_h^t \colon M \to M$.*

Proof (Barbot).[1] We use the formulation of hyperbolicity in terms of suitable Lyapunov–Lorentz metrics as described in Proposition A.1; this is a reformulation of the usual cone criterion for hyperbolicity.

By Proposition A.1 there is a pair of Lyapunov–Lorentz metrics for φ^t. We deform these to work as needed for φ_h^t. First, we arrange (using a partition of unity, say) for the Lyapunov–Lorentz metrics for φ^t to have the form

$$Q^\pm = \pm dw\,ds - c\,dt^2$$

in Λ, where c is chosen sufficiently small to ensure that the positive $Q^\pm$-cone contains $E^\pm$. With λ as in Figure 3 set

$$\beta(t) := \frac{\int_{-\eta}^{t} \lambda(x)\,dx}{\int_{-\eta}^{\eta} \lambda(x)\,dx},$$

i.e., β is smooth, nondecreasing, 0 near $-\eta$ and 1 near η.

CLAIM 4.4. *Taking $Q_0^\pm$ and $Q_1^\pm$ to be the old Lyapunov–Lorentz metrics outside Λ and*

$$Q_i^\pm := \pm\big(dw\,ds + (\beta(t) + i)f'(w)\,dw^2\big) - c\,dt^2$$

inside defines Lyapunov–Lorentz metrics for φ_h^t. Here, $i = 0$ on one side of the surgery and $i = 1$ on the other.

Proof. Our choice of f and β ensures that these are smooth metrics.

These choices fit together, i.e., F sends the choice on one side to that on the other:

$$\begin{aligned} F_* Q_0^\pm &= F_*\big(\pm(dw\,ds + \beta(t)f'(w)\,dw^2) - c\,dt^2\big) \\ &= \pm(dw\,d(s + f(w)) + \beta(t)f'(w)\,dw^2) - c\,dt^2 \\ &= \pm(dw\,ds + f'(w)\,dw\,dw + \beta(t)f'(w)\,dw^2) - c\,dt^2 = Q_1^\pm. \end{aligned}$$ □

Of the required properties in Proposition A.1, 2. and 3. are clear. Since on Λ the flow is represented by a shift in time, φ_h^t is at worst a $Q_h^\pm$-isometry and in general changes $Q_h^\pm$ by $\pm f'(w)\,dw^2$, which goes in the "right" direction either way since $f' \geq 0$. Since $\varphi = \varphi_h$ outside of Λ, properties 1. and 4. in Proposition

[1] Thierry Barbot kindly allowed us to reproduce here the elegant and hitherto unpublished version [4, page 54] of the proofs in [22, 24].

A.1 are inherited (with the same constants even) from the same properties for φ^t. □

5. CONSTRUCTING EXAMPLES

In this section we describe how our construction subsumes those of Handel and Thurston [24] and Goodman [22] and goes beyond these by considering different initial geodesics as starting points or applying surgeries simultaneously or repeatedly.

5.1. **Handel–Thurston surgery and beyond.** To reproduce the setting used by Handel and Thurston, consider a negatively curved surface and select a closed geodesic $\mathfrak{c}\colon S^1 \to \Sigma$, $s \mapsto \mathfrak{c}(s)$. If this geodesic is simple and separating, we denote the unit tangent bundles of the two components of the surface by M_1 and M_2.

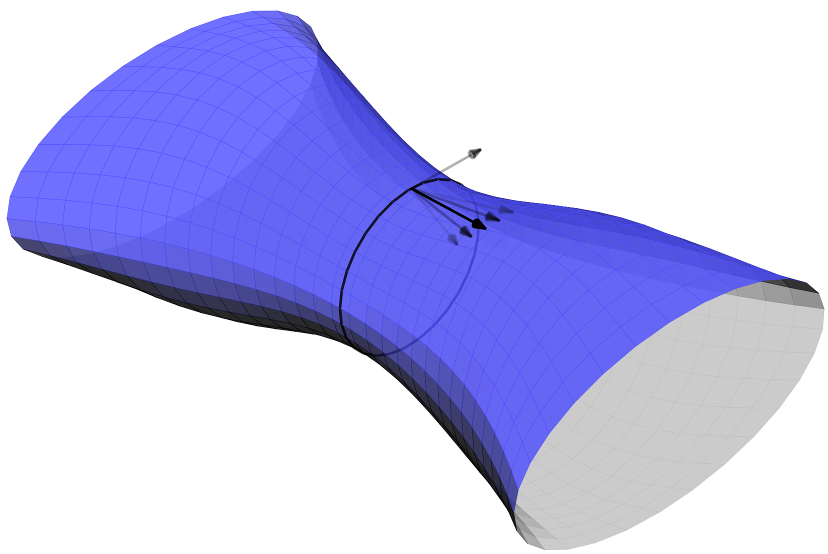

FIGURE 4. A simple closed geodesic and normal vectors

We deform the geodesic flow g^t on the unit tangent bundle $S\Sigma$ of the surface to a flow on a new 3-manifold. The common boundary of M_1 and M_2 is a torus $S^1 \times S^1$ parametrized by the parameter s of the geodesic $\mathfrak{c}$ and the angle θ with the tangent vector of the geodesic. The Legendrian knot is the unit vector field perpendicular to $\mathfrak{c}$ given by $\theta = -\pi/2$.

Although we localize the surgery in an annulus around $\theta = -\pi/2$ inside the torus, topologically this is clearly the same as in [24].

To parametrize a neighborhood Λ of the perpendicular unit vector field γ of $\mathfrak{c}$, linearize the angle θ with the tangent vector field to $\mathfrak{c}$ by taking $w := \dfrac{\ell}{2\pi}\cos\theta$ for θ near $-\pi/2$, where ℓ is the length of $\mathfrak{c}$. This gives parameters

$$(t, s, w) \in \Omega := (-\eta, \eta) \times S^1 \times (-\epsilon, +\epsilon), \tag{3}$$

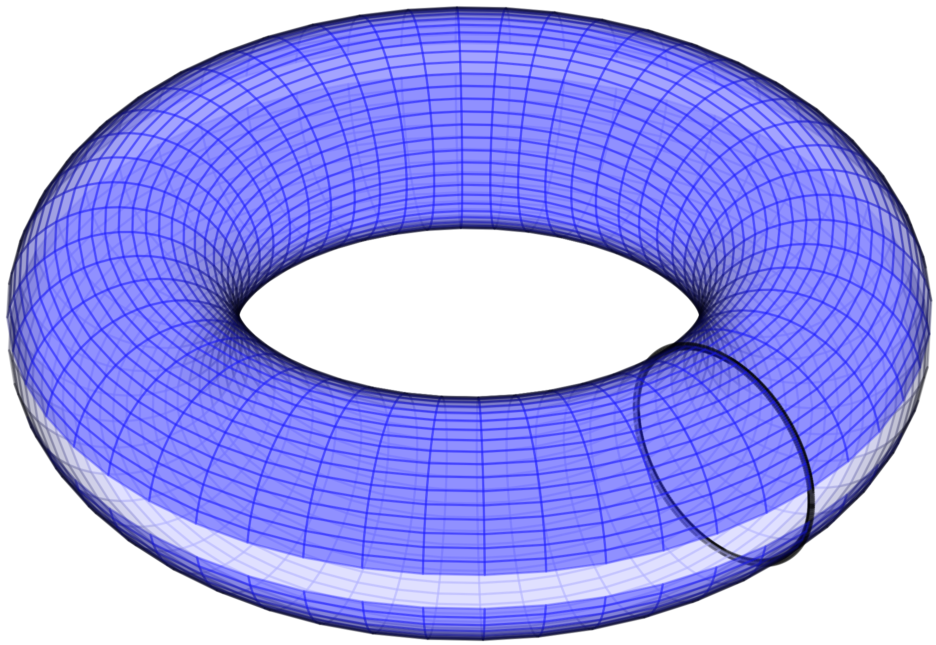

FIGURE 5. The annulus $S^1 \times (-\pi/2 - \epsilon, -\pi/2 + \epsilon) \subset S^1 \times S^1$ before surgery

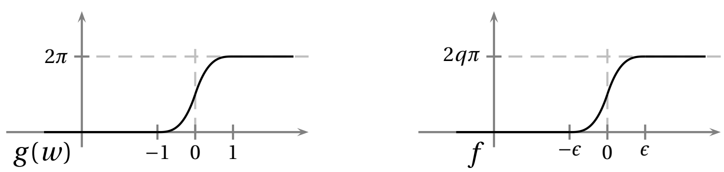

FIGURE 6. The functions g and f

where $t \in (-\eta, \eta)$ parametrizes the flow direction and $s \in S^1$ is the parameter along $\mathfrak{c}$. γ is parametrized by $\{0\} \times S^1 \times \{0\}$. This gives a chart as in Proposition 2.2:

LEMMA 5.1. *The standard contact form A in this chart satisfies*

$$A = dt + w\,ds, \quad dA = dw \wedge ds \quad \text{and} \quad A \wedge dA = dt \wedge dw \wedge ds.$$

Proof. If g_s denotes the Riemannian metric at $(0, s) \in \Sigma$ and we write $(0, s, \theta) = (x, u) \in S\Sigma$, then for a vector

$$Z = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} + c\frac{\partial}{\partial \theta}$$

we have

$$A_{(0,s,\theta)}(Z) = g_s(u, d\pi(Z)) = g_s(u, a\,d\pi(\frac{\partial}{\partial t}) + b\,d\pi(\frac{\partial}{\partial s})) = a + b g_s(u, \frac{\partial}{\partial s})$$

since $d\pi(\partial/\partial\theta) = 0$.

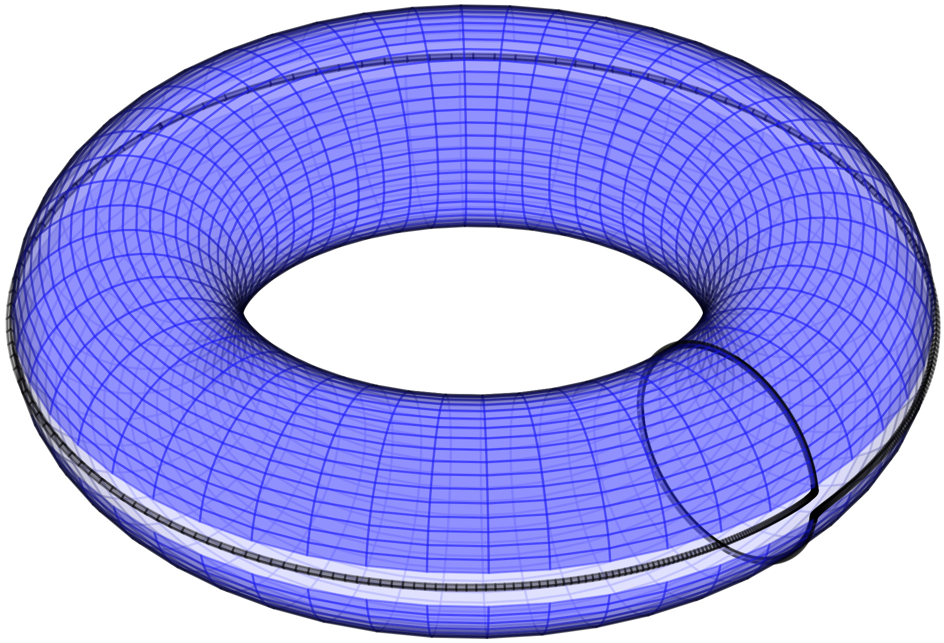

FIGURE 7. The annulus after surgery ($q = 1$)

Taking S^1 to have length 2π we necessarily obtain $\|\partial/\partial s\| = \ell/2\pi$. Since a priori $A_{(t,s,\theta)} = dt + g\,ds + h\,d\theta$ with functions g and h, this implies $A_{(t,s,\theta)} = dt + \frac{\ell}{2\pi}\cos\theta\,ds$, i.e., $A = dt + w\,ds$. The other claims immediately follow. □

While the Handel–Thurston surgery is described for a separating geodesic and hyperbolicity of the phase space is obtained below using a filling geodesic, neither of these is needed for the perpendicular vector field to the geodesic to define a Legendrian knot; our surgery subsumes all these possibilities. Moreover, such surgeries can be carried out successively, using Legendrian knots from the preceding intermediate stage.

5.2. **Goodman surgery and beyond.** Goodman [22] gave surgeries inspired by those of Handel and Thurston but carried out in the vicinity of orbits (Figure 8, taken from [22], shows her Dehn torus as seen along the orbit in question). In the contact category, a natural starting point for Goodman's surgery is a geodesic flow, where our construction subsumes hers because we can isotope orbits to Legendrian knots:

THEOREM 5.2. *Consider the geodesic flow on the unit tangent bundle of a negatively curved surface. Then for each finite union of free homotopy classes of closed curves there exists a transverse Legendrian link that includes precisely one knot in each of the chosen homotopy classes.*

Proof. Since negative curvature implies hyperbolicity of the geodesic flow, each of the chosen homotopy classes contains a closed orbit of the geodesic flow.

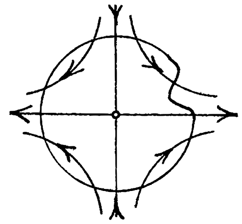

FIGURE 8. The Goodman surgery annulus (tilted segment in first quadrant)

These are pairwise disjoint, and each of these is isotopic to a knot given by rotation of the tangent vector field by $-\pi/2$, and these knots are pairwise disjoint. As before, they are Legendrian knots. □

After choosing pairwise disjoint neighborhoods of these Legendrian knots, we can perform surgery in each of these as described in Section 2.

We remark that on one hand Goodman's surgery (unlike that presented by Handel and Thurston) can be carried out recursively; it applies near any closed orbit of any Anosov flow. On the other hand, the Anosov flows she obtained on hyperbolic manifolds have a suspension of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ as a starting point, which is far from contact.

6. GEOMETRY OF THE HANDEL–THURSTON SURGERY

We return to the Handel–Thurston context of Subsection 5.1 and show that in this case we get flows that are not topologically orbit equivalent to an algebraic flow.

DEFINITION 6.1. A 3-manifold is said to be Seifert-fibered if it admits a decomposition into a disjoint union of circles (the fibers) such that each fiber has a tubular neighborhood diffeomorphic (in a fiber-preserving way) to the torus $D^2 \times S^1$ obtained from $D^2 \times [0,1]$ by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ via a rational rotation.

We now recall the proof Handel and Thurston [24] that the flows thus obtained are not topologically orbit equivalent to an algebraic flow.

THEOREM 6.2. *When the surgery is carried out using a separating curve, the flow φ_h^t in Theorem 4.3 is not topologically orbit equivalent to an algebraic Anosov flow. More strongly* [44, page 419], *no finite cover of the surgered manifold M from Definition 3.1 is a Seifert-fibered manifold (much less a sphere bundle) or a torus bundle over a circle.*

Proof. ([24]) We study finite covers of M by examining their fundamental group. The two pertinent facts are

1. The fundamental group of a torus bundle over a circle is solvable; thus we wish to show that $\pi_1(M)$ is not virtually solvable, i.e., has no solvable finite-index subgroup.

2. The fundamental group of a Seifert-fibered manifold contains an infinite normal cyclic subgroup generated by a regular fiber [39, page 432]; thus we want to show that no finite-index subgroup of $\pi_1(M)$ contains an infinite cyclic normal subgroup.

One observation that we will use for both of these items is the following.

REMARK 6.3. If a group contains a finite-index subgroup H and a free subgroup F, then H contains a subgroup of F that is isomorphic to F.

By the van Kampen Theorem [27, Theorem 1.20], we have

$$\pi_1(M) = \pi_1(M_1) \underset{\pi_1(\partial M_1)}{*} \pi_1(M_2) = \pi_1(M_1) \underset{\pi_1(\partial M_2)}{*} \pi_1(M_2),$$

using the isomorphism $F_*\colon \pi_1(\partial M_1) \to \pi_1(\partial M_2)$ induced by F (as introduced in Definition 3.1).

Puncturing a surface of genus g and retracting the remainder to its skeleton (a string of $2g$ circles) shows that the fundamental group is a free group with $2g$ generators. Thus, we see that $\pi_1(M_i) = F_i \oplus \mathbb{Z}$ for $i = 1,2$, where F_1 and F_2 are free groups.

If $H < \pi_1(M)$ has finite index, then, as remarked above, it contains a free group inherited from F_1 or F_2, and since this holds recursively, H is not solvable, and item 1. above is settled.

To settle item 2. suppose $\langle g\rangle < H$ is an infinite cyclic normal subgroup. This means that for every $h \in H$ there is a $p_h \in \mathbb{Z}$ such that $hgh^{-1} = g^{p_h}$. Clearly, $p_{h_1h_2} = p_{h_1}p_{h_2}$ for any $h_1, h_2 \in H$, so $p_{\mathrm{Id}} = 1$ implies that for each $h \in H$ we have $p_h \in \{\pm 1\}$ and $p_h = p_{h^{-1}}$. Thus, after possibly passing to the index-2 subgroup $\{h \in H \mid p_h = 1\}$, we may assume without loss of generality that H is in the centralizer of g, i.e., $gh = hg$ for all $h \in H$.

As remarked above, there is a free group $F_H \subset F_1 \cap H$ isomorphic to F_1 (we only need that it is large enough). We can write $g = wr^k$ with w a word in generators of F_1 and F_2 only and r the generator of the S^1-factor of $\pi_1(M_1)$: writing one of the generators of $\pi_1(\partial M_2)$ as $\alpha_a r = a = \omega_a {r'}^k$ with $\alpha_a \in F_1$, $\omega_a \in F_2$ and r' the generator of the S^1-factor of $\pi_1(M_2)$, we find that any occurrence of $r\omega$ with $\omega \in F_2$ can be rewritten as $\alpha_a^{-1}\omega_a\omega\omega_a^{-1}\alpha_a r$; one applies this recursively to get the claim. We thus find that for any $h \in F_H \subset F_1$ we get

$$whr^k = wr^k h = gh = hg = hwr^k,$$

i.e., $wh = hw$. But for $w \neq \mathrm{Id}$, this only holds when h is a power of w. Since F_H is not a subgroup of a cyclic group, we must have $g = r^k$ for some $k \in \mathbb{Z}$.

However, the same reasoning using F_2 shows that $g = {r'}^\ell$ for some $\ell \in \mathbb{Z}$, where r' is the generator of the S^1-factor of $\pi_1(M_2)$. This is incompatible with the earlier observation that $g = r^k \sim (sr')^k$, where s represents the word in F_2 corresponding to the slope of the surgery—unless $s = \mathrm{Id}$ and the surgery is therefore trivial.

This implies item 2. above. □

APPENDIX A. ANOSOV FLOWS AND LYAPUNOV–LORENTZ METRICS

This appendix reproduces a way of expressing the Alexeev cone criterion for hyperbolicity in terms of Lorentz metrics that behave analogously to Lyapunov functions or metrics [4, page 16].

PROPOSITION A.1. *A smooth flow $\varphi^t\colon M \to M$ of a 3-manifold M is an Anosov flow if and only if there are two continuous Lorentz metrics Q^+ and Q^- on M and constants $a, b, c, T > 0$ such that*

1. *for all $v \in T_xM$, $t > T$, if $Q^\pm(v) > 0$ then $Q^\pm(D_x\varphi^{\pm t}(v)) > ae^{bt}Q^\pm(v)$,*
2. *$C^+ \cap C^- = \varnothing$, where $C^\pm$ is the $Q^\pm$-positive cone,*
3. *$Q^\pm(X) = -c$ where X is the generating vector field,*
4. *$D_x\varphi^{\pm T}(\overline{C^\pm(x)}) \smallsetminus \{0\} \subset C^\pm(\varphi^{\pm T}(x))$.*

Proof. If φ^t is an Anosov flow we can choose disjoint cones around the strong stable and unstable directions, neither of which contains X. These define (up to a factor) the Lorentz metrics, and choosing $c = 1$ fixes the metrics; we omit the details.

Assume now the above conditions for two continuous Lorentz metrics $Q^\pm$ and constants $a, b, c, T > 0$. The cone fields $C^\pm$ induce fields $\mathscr{E}^\pm$ of ellipses in the projectivization PTM of TM, and φ^t acts on fields of ellipses by $(\varphi^t_*\mathscr{E})(x) := PD_{\varphi^{-t}(x)}\varphi^t(\mathscr{E}(\varphi^{-t}(x)))$. Then

- condition 2. implies that $\mathscr{E}^+_t(x) \cap \mathscr{E}^-_t(x) = \varnothing$,
- condition 4. implies that $\overline{\mathscr{E}^\pm_T(x)} \subset \operatorname{int}\mathscr{E}^\pm(x)$.

If we endow each $\mathscr{E}^\pm(x)$ with the Hilbert metric then this last property (strict nesting) implies that $D\varphi^{\pm T}$ induces contractions $\mathscr{E}^\pm(x) \to \mathscr{E}^\pm(\varphi^{\pm T}(x))$ of the Hilbert metrics with a factor that can be chosen uniformly by compactness of M. Thus, the diameter of $\mathscr{E}^\pm_t(x) \subset \mathscr{E}^\pm(x)$ as measured by the Hilbert metric on $\mathscr{E}^\pm(x)$ goes to 0 exponentially, so $\Delta^\pm(x) := \bigcap_{t>T}\mathscr{E}^\pm_t(x)$ are points, and $\Delta^+(x) \neq \Delta^-(x)$ for all $x \in M$ since $\mathscr{E}^+_t(x) \cap \mathscr{E}^-_t(x) = \varnothing$.

Clearly $\Delta^\pm$ define φ^t-invariant line fields $E^\pm$, and since $X \notin C^\pm$ by condition 3., $\Delta^+(x) \neq X(x) \neq \Delta^-(x)$.

Now choose a continuous Riemannian metric on M whose unit spheres intersect $E^\pm$ in points for which $Q^\pm = 1$. Then condition 1. implies that $E^\pm$ are exponentially expanding and contracting, respectively, as required. □

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