Andrews–Curtis Groups and the Andrews–Curtis Conjecture

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Credits and further info

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cf. Grigorchuk and Kurchanov (1998)

Seminar Plan

1. From Transformations to Automorphisms
2. The Main Theorem and Consequences
3. Sketch Proof

From Transformations to Automorphisms

Andrews–Curtis Transformations

Let $X = \{x_1, x_2, \dots, x_n\}$ for some $n \geq 2$. The following transformations on the set

$$\{(u_1, \dots, u_m) \mid u_1, \dots, u_m \in \mathrm{F}(X)\}$$

are known as *elementary Andrews–Curtis transformations (of rank (n, m))*:

$AC1(i, j)$ Replace u_i by $u_i u_j$

$AC2(i)$ Replace u_i by u_i^{-1}

$AC3(i, k, +)$ Replace u_i by $x_k^{-1} u_i x_k$

$AC3(i, k, -)$ Replace u_i by $x_k u_i x_k^{-1}$.

AC Transformations II

If τ is an elementary AC transformation and

$$(u_1, u_2, \ldots, u_m) \stackrel{\tau}{\longmapsto} (v_1, v_2, \ldots, v_m),$$

then $\langle\langle u_1, u_2, \ldots, u_m \rangle\rangle^{F(X)} = \langle\langle v_1, v_2, \ldots, v_m \rangle\rangle^{F(X)}$.

In case $m = n$, if $\exists \tau_1, \ldots, \tau_t$

$$(x_1, x_2, \ldots, x_n) \stackrel{\tau_1}{\longmapsto} \ldots \stackrel{\tau_t}{\longmapsto} (s_1, s_2, \ldots, s_n)$$

then $\langle x_1, \ldots, x_n \mid s_1, \ldots, s_n \rangle$ defines the trivial group.

The Andrews–Curtis Conj. (1965)

Conjecture 1 *Let* $s_1, \ldots, s_n \in \mathrm{F}(X)$*. The presentation*

$$\langle x_1, \ldots, x_n \mid s_1, \ldots, s_n \rangle$$

defines the trivial group if and only if there exists a finite sequence of elementary Andrews–Curtis transformations carrying $(x_1, \ldots, x_n)$ *to* $(s_1, \ldots, s_n)$.

Why do Topologists Care?

Corollary 1 (Andrews–Curtis) *Let K be a finite contractible 2-dimensional subcomplex of a combinatorial 5-manifold M, and let N be any regular neighborhood of K in M. If the conjecture is true, then N is a 5-cell.*

The group $\mathfrak{A}C(n, m)$

A transformation on the set

$$\{(u_1, \ldots, u_m) \mid u_1, \ldots, u_m \in \mathrm{F}(X)\}$$

which may be achieved by a finite sequence of elementary Andrews–Curtis transformations is called an *Andrews–Curtis transformation* (of rank (n, m)).

The set of Andrews–Curtis transformations of rank (n, m) is a group $\mathfrak{A}C(n, m)$ under the operation of concatenation.

Formal AC Transformations

Let $R = \{r_1, r_2, \ldots, r_m\}$ for some $m \geq 1$. The following transformations on the set

$$\{(v_1, \ldots, v_{n+m}) \mid v_1, \ldots, v_{n+m} \in \mathrm{F}(X \cup R)\}$$

are known as *elementary formal Andrews–Curtis transformations (of rank* (n, m)*)*:

$AC1_{\mathrm{f}}(i, j)$	Replace v_{n+i} by $v_{n+i}v_{n+j}$
$AC2_{\mathrm{f}}(i)$	Replace v_{n+i} by v_{n+i}^{-1}
$AC3_{\mathrm{f}}(i, k, +)$	Replace v_{n+i} by $v_k^{-1}v_{n+i}v_k$
$AC3_{\mathrm{f}}(i, k, -)$	Replace v_{n+i} by $v_k v_{n+i} v_k^{-1}$.

The group $\mathfrak{A}C_{\mathrm{f}}(n, m)$

A transformation on the set

$$\{(v_1, \ldots, v_{n+m}) \mid v_1, \ldots, v_{n+m} \in \mathrm{F}(X \cup R)\}$$

which may be achieved by a finite sequence of elementary formal Andrews–Curtis transformations is called a *formal Andrews–Curtis transformation* (of rank (n, m)).

The set of formal Andrews–Curtis transformations of rank (n, m) is a group $\mathfrak{A}C_{\mathrm{f}}(n, m)$ under the operation of concatenation.

Example

$$
\begin{aligned}
&(u_1, u_2)\\
&\downarrow^{AC3(1,2,+)}\\
&(x_2^{-1} u_1 x_2, u_2)\\
&\downarrow^{AC1(1,2)}\\
&(x_2^{-1} u_1 x_2 u_2, u_2)\\
&\downarrow^{AC2(2)}\\
&(x_2^{-1} u_1 x_2 u_2, u_2^{-1}).
\end{aligned}
$$

Example

$$
\begin{array}{l|l}
(u_1, u_2) & (v_1, v_2, \; v_3, v_4) \\
\downarrow^{AC3(1,2,+)} & \qquad\quad \downarrow^{AC3_{\mathrm{f}}(1,2,+)} \\
(x_2^{-1}u_1x_2, u_2) & (v_1, v_2, \; v_2^{-1}v_3v_2, v_4) \\
\downarrow^{AC1(1,2)} & \qquad\quad \downarrow^{AC1_{\mathrm{f}}(1,2)} \\
(x_2^{-1}u_1x_2u_2, u_2) & (v_1, v_2, \; v_2^{-1}v_3v_2v_4, v_4) \\
\downarrow^{AC2(2)} & \qquad\quad \downarrow^{AC2_{\mathrm{f}}(2)} \\
(x_2^{-1}u_1x_2u_2, u_2^{-1}) & (v_1, v_2, \; v_2^{-1}v_3v_2v_4, v_4^{-1}).
\end{array}
$$

Example

$$
\begin{array}{l|l}
(u_3, u_4) & (v_1, v_2,\ \ v_3, v_4) \\
\big\downarrow^{AC3(1,2,+)} & \qquad\quad \big\downarrow^{AC3_{\mathrm{f}}(1,2,+)} \\
(x_2^{-1} u_3 x_2, u_4) & (v_1, v_2,\ \ v_2^{-1} v_3 v_2, v_4) \\
\big\downarrow^{AC1(1,2)} & \qquad\quad \big\downarrow^{AC1_{\mathrm{f}}(1,2)} \\
(x_2^{-1} u_3 x_2 u_4, u_4) & (v_1, v_2,\ \ v_2^{-1} v_3 v_2 v_4, v_4) \\
\big\downarrow^{AC2(2)} & \qquad\quad \big\downarrow^{AC2_{\mathrm{f}}(2)} \\
(x_2^{-1} u_3 x_2 u_4, u_4^{-1}) & (v_1, v_2,\ \ v_2^{-1} v_3 v_2 v_4, v_4^{-1}).
\end{array}
$$

Example

$$
\begin{array}{ll|ll}
\langle x_1, x_2 \mid & x_1, x_2 \rangle & (x_1, x_2, & r_1, r_2) \\
& \downarrow^{AC3(1,2,+)} & & \downarrow^{AC3_{\mathrm{f}}(1,2,+)} \\
\langle x_1, x_2 \mid & x_2^{-1} x_1 x_2, x_2 \rangle & (x_1, x_2, & x_2^{-1} r_1 x_2, r_2) \\
& \downarrow^{AC1(1,2)} & & \downarrow^{AC1_{\mathrm{f}}(1,2)} \\
\langle x_1, x_2 \mid & x_2^{-1} x_1 x_2^2, x_2 \rangle & (x_1, x_2, & x_2^{-1} r_1 x_2 r_2, r_2) \\
& \downarrow^{AC2(2)} & & \downarrow^{AC2_{\mathrm{f}}(2)} \\
\langle x_1, x_2 \mid & x_2^{-1} x_1 x_2^2, x_2^{-1} \rangle & (x_1, x_2, & x_2^{-1} r_1 x_2 r_2, r_2^{-1}).
\end{array}
$$

Example Cont.

In case $m = n$, let $\epsilon : \mathrm{F}(X \cup R) \rightarrow \mathrm{F}(X)$ denote the map defined by

$$(x_1, \ldots, x_n, r_1, \ldots, r_n) \mapsto (x_1, \ldots, x_n, x_1, \ldots, x_n).$$

Note that

$$\epsilon(x_1, x_2, x_2^{-1} r_1 x_2 r_2, r_2^{-1}) = (x_1, x_2, x_2^{-1} x_1 x_2^2, x_2^{-1}).$$

Some relationships b/w groups

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The group $\mathfrak{N}(n+m)$ of Nielsen transformations of rank $(n+m)$ is anti-isomorphic to $\mathrm{Aut}\,\mathrm{F}(X \cup R)$.

Write $\mathrm{AC}(\mathrm{n},\mathrm{m})$ for the image of $\mathfrak{A}C_{\mathrm{f}}(n,m)$ in $\mathrm{Aut}\,\mathrm{F}(X \cup R)$.

Notation for automorphisms

We introduce notation for some elements of $\operatorname{Aut} \mathrm{F}(X \cup R)$ as follows

$$
\begin{aligned}
(AC1(i,j) &\mapsto) \quad \mu_{ij} : r_i \mapsto r_i r_j \\
(AC2(i) &\mapsto) \quad \sigma_i : r_i \mapsto r_i^{-1} \\
(AC3(i,k,+) &\mapsto) \quad \chi_{ik} : r_i \mapsto x_k^{-1} r_i x_k \\
(AC3(i,k,-) &\mapsto) \quad \chi_{ik}^{-1} : r_i \mapsto x_k r_i x_k^{-1}.
\end{aligned}
$$

So $\mathrm{AC}(\mathrm{n}, \mathrm{m})$ is generated by

$$\{\mu_{i,j}, \sigma_i, \chi_{i,k} \mid 1 \leq k \leq n, 1 \leq i,j \leq m, i \neq j\}.$$

The anti-isomorphism

If $\tau_1, \tau_2, \ldots, \tau_p \in \mathfrak{N}(n+m)$ and α denotes the anti-isomorphism $\mathfrak{N}(n+m) \longrightarrow \mathrm{Aut}\, F(X \cup R)$, then

$$\tau_p \ldots \tau_2 \tau_1 (x_1, \ldots, x_n, r_1, \ldots, r_m) = \alpha(\tau_1)\alpha(\tau_2)\ldots\alpha(\tau_p)(x_1, \ldots, x_n, r_1, \ldots, r_m).$$

(On LHS we perform transformations on $(n+m)$ tuples, on RHS we perform substitution inside each entry.)

Example Cont.

$$\left(x_1, x_2, \;\; r_1, r_2\right)$$

$$\big\downarrow^{AC3_f(1,2,+)}$$

$$\left(x_1, x_2, \;\; x_2^{-1} r_1 x_2, r_2\right)$$

$$\big\downarrow^{AC1_f(1,2)}$$

$$\left(x_1, x_2, \;\; x_2^{-1} r_1 x_2 r_2, r_2\right)$$

$$\big\downarrow^{AC2_f(2)}$$

$$\left(x_1, x_2, \;\; x_2^{-1} r_1 x_2 r_2, r_2^{-1}\right)$$

Example Cont.

$$\begin{array}{l|l}
\begin{array}{l}
(x_1, x_2, \quad r_1, r_2) \\
\qquad\qquad \downarrow^{AC3_{\mathrm{f}}(1,2,+)} \\
(x_1, x_2, \quad x_2^{-1} r_1 x_2, r_2) \\
\qquad\qquad \downarrow^{AC1_{\mathrm{f}}(1,2)} \\
(x_1, x_2, \quad x_2^{-1} r_1 x_2 r_2, r_2) \\
\qquad\qquad \downarrow^{AC2_{\mathrm{f}}(2)} \\
(x_1, x_2, \quad x_2^{-1} r_1 x_2 r_2, r_2^{-1})
\end{array} &
\begin{array}{l}
(x_1, x_2, \quad r_1, r_2) \\
\qquad\qquad \downarrow^{\sigma_2} \\
(x_1, x_2, \quad r_1, r_2^{-1}) \\
\qquad\qquad \downarrow^{\mu_{12}} \\
(x_1, x_2, \quad r_1 r_2, r_2^{-1}) \\
\qquad\qquad \downarrow^{\chi_{12}} \\
(x_1, x_2, \quad x_2^{-1} r_1 x_2 r_2, r_2^{-1}).
\end{array}
\end{array}$$

How the groups fit together

$$\begin{array}{ccccc} \mathfrak{A}C(n,m) & \cong & \mathfrak{A}C_{\mathrm{f}}(n,m) & \overset{a}{\cong} & \mathrm{AC}(\mathrm{n},\mathrm{m}) \\ & & \downarrow & & \downarrow \\ & & \mathfrak{N}(n+m) & \overset{a}{\cong} & \mathrm{Aut}\,\mathrm{F}(X \cup R) \end{array}$$

Transformations on $\{(u_1,\ldots,u_m) \mid u_i \in \mathrm{F}(X)\}$

Transformations on $\{(v_1,\ldots,v_{n+m}) \mid v_i \in \mathrm{F}(X \cup R)\}$

Automorphisms of $\mathrm{F}(X \cup R)$

Reformulation of the AC Conj.

Conjecture 2 *Let* $s_1, \ldots, s_n \in \mathrm{F}(X)$. *Then* $\langle x_1, \ldots, x_n \mid s_1, \ldots, s_n \rangle$ *defines the trivial group if and only if there exists* $\phi \in \mathrm{AC}(\mathrm{n}, \mathrm{n})$ *such that*

$$\epsilon \circ \phi(r_1, \ldots, r_n) = (s_1, \ldots, s_n).$$

Another Reformulation

Define

$$\mathcal{P}T(n,m) := \{\phi(r_1,\ldots,r_m) \mid \phi \in \mathrm{AC(n,m)}\}.$$

Conjecture 3 *Let* $s_1,\ldots,s_n \in \mathrm{F}(X)$. *Then*

$$\langle x_1,\ldots,x_n \mid s_1,\ldots,s_n\rangle$$

defines the trivial group if and only if

$$\epsilon^{-1}(s_1,\ldots,s_n) \cap \mathcal{P}T(n,n) \neq \emptyset.$$

The Generalized Word Problem

The Main Theorem

Theorem 1 *Let* $\mathrm{pr}_X : \mathrm{F}(X \cup R) \to \mathrm{F}(X)$ *be the homomorphism defined by*

$$(x_1, \ldots, x_n, r_1, \ldots, r_m) \mapsto (x_1, \ldots, x_n, 1, \ldots, 1).$$

An m*-tuple* $(w_1, \ldots, w_m) \in \mathrm{F}(X \cup R)^m$ *is an element of* $\mathcal{P}T(n, m)$ *if and only if the following properties both hold:*

(1) $(x_1, \ldots, x_n, w_1, \ldots, w_m)$ *is a basis for* $\mathrm{F}(X \cup R)$

(2) $\mathrm{pr}_X(w_1, \ldots, w_m) = (1, \ldots, 1)$.

Some Consequences

Corollary 2 *The set* $\mathcal{P}T(n, m)$ *is a recursive subset of* $\mathrm{F}(X \cup R)^m$.

Corollary 3 (The generalized word problem) *For each* $\phi \in \mathrm{Aut}\,\mathrm{F}(X \cup R)$, *it is decidable whether or not* $\phi \in \mathrm{AC}(\mathrm{n}, \mathrm{m})$.

Corollary 4 *In case* $m = n$, *the intersection*

$$\epsilon^{-1}(s_1, \ldots, s_n) \cap \mathcal{P}T(n, n)$$

is a recursive subset of $\mathrm{F}(X \cup R)^n$.

Some Consequences II

Corollary 5 *The set*

$$\left\{(s_1,\ldots,s_n) \mid \langle x_1,\ldots,x_n \,|\, s_1,\ldots,s_n\rangle \text{ is an } \right.$$

$$\left. \text{AC trivializable presentation of the trivial group} \right\}$$

is the homomorphic image of a recursive set.

Sketch Proof

More Notation

Write $\psi_{ik} : r_i \mapsto x_k^{-1} r_i$.

Write

$$\begin{aligned} \mathrm{St}(X) := \{\phi \in \mathrm{Aut}\,\mathrm{F}(X \cup R) \mid \\ \phi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)\}. \end{aligned}$$

Note that $\mathrm{AC(n,m)} \leq \mathrm{St}(X) \leq \mathrm{Aut}\,\mathrm{F}(X \cup R)$.

Stabilizers

Using McCool . . .

Lemma 1 *Each element of* $\mathrm{St}(X)$ *may be written as a product in the alphabet*

$$\{\mu_{i,j}, \sigma_i, \chi_{i,k}, \chi_{i,k}^{-1}, \psi_{i,k}, \psi_{i,k}^{-1} \mid 1 \le k \le n, 1 \le i, j \le m, i \ne j\}.$$

The Easy Direction

The following is evident in Metzler . . .

If $(w_1, \ldots, w_m) \in \mathcal{P}T(n, m)$, then the following both hold:

(1) $(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is a basis for $\mathrm{F}(X \cup R)$

(2) $\mathrm{pr}_X(w_1, \ldots, w_m) = (1, \ldots, 1)$.

(Proof by Induction.)

The Other Direction

Suppose the following both hold:

(1) $(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is a basis for $\mathrm{F}(X \cup R)$

(2) $\mathrm{pr}_X(w_1, \ldots, w_m) = (1, \ldots, 1)$.

The Other Direction

Suppose the following both hold:

(1) $(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is a basis for $\mathrm{F}(X \cup R)$

(2) $\mathrm{pr}_X(w_1, \ldots, w_m) = (1, \ldots, 1)$.

Then there exists $\phi \in \mathrm{St}(X)$ such that

$$\phi(x_1, \ldots, x_n, r_1, \ldots, r_m) = (x_1, \ldots, x_n, w_1, \ldots, w_m).$$

The Other Direction II

By Lemma 1 there exists a word W in the alphabet

$$\{\mu_{i,j}, \sigma_i, \chi_{i,k}, \chi_{i,k}^{-1}, \psi_{i,k}, \psi_{i,k}^{-1} \mid 1 \leq k \leq n, 1 \leq i,j \leq m, i \neq j\}$$

such that W spells ϕ.

Our task is to rewrite W as a word W' in the alphabet

$$\{\mu_{i,j}, \sigma_i, \chi_{i,k}, \chi_{i,k}^{-1} \mid 1 \leq k \leq n, 1 \leq i,j \leq m, i \neq j\}.$$

The Other Direction III

We have that there exist Nielsen transformations $\tau_1, \tau_2, \ldots, \tau_t$ in the alphabet

$$\alpha^{-1}\{\mu_{i,j}, \sigma_i, \chi_{i,k}, \chi_{i,k}^{-1}, \psi_{i,k}, \psi_{i,k}^{-1} \mid 1 \leq k \leq n, 1 \leq i, j \leq m, i \neq j\}$$

such that

$$\tau_t \ldots \tau_2 \tau_1 (x_1, \ldots, x_n, r_1, \ldots, r_m) = (x_1, \ldots, x_n, w_1, \ldots, w_n).$$

The Other Direction IV

Replace each τ_i by $\mu_i \in \mathfrak{A}\mathrm{C}_{\mathrm{f}}(n, m)$ such that:
If

$$\tau_i \ldots \tau_2\tau_1(x_1, \ldots, x_n, r_1, \ldots, r_m) = (x_1, \ldots, x_n, u_1, \ldots, u_m)$$

then

$$\mu_i \ldots \mu_2\mu_1(x_1, \ldots, x_n, r_1, \ldots, r_m) = (x_1, \ldots, x_n, a_1u_1, \ldots, a_mu_m)$$

for some $a_1, \ldots, a_m \in \mathrm{F}(X)$.

The Other Direction V

So

$$\mu_t \ldots \mu_2\mu_1(x_1, \ldots, x_n, r_1, \ldots, r_m) = \\ (x_1, \ldots, x_n, a_1 w_1, \ldots, a_m w_m)$$

for some $a_1, \ldots, a_m \in \mathrm{F}(X)$.

The Other Direction V

So

$$\mu_t \dots \mu_2 \mu_1 (x_1, \dots, x_n, r_1, \dots, r_m) = (x_1, \dots, x_n, a_1 w_1, \dots, a_m w_m)$$

for some $a_1, \dots, a_m \in \mathrm{F}(X)$.

Since $\mu_1, \dots, \mu_t \in \mathfrak{A}\mathrm{C}_{\mathrm{f}}(n, m)$, $\mathrm{pr}_X(a_i w_i) = 1$.

But $\mathrm{pr}_X(w_i) = 1$, so $\mathrm{pr}_X(a_i) = 1$.

And $a_i \in \mathrm{F}(X)$, so $\mathrm{pr}_X(a_i) = a_i = 1$.