Understanding when the outer automorphism group of a right-angled Coxeter group is finite

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Credits

This material is joint work with Mauricio Gutierrez and Kim Ruane.

A preprint is available at

`http://www.tufts.edu/~apiggo01`

Right-angled Coxeter groups

Let

- Γ a finite graph with no circuits of length < 3.
- V the vertices of Γ.

Write $W = W(\Gamma)$ for the group presented by

$$\langle V \mid v^2 \ \ (v \in V), (uw)^2 \ \ (u, w \in V \text{ and adjacent in } \Gamma)\rangle$$

Rigidity: The isomorphism class of W determines the isomorphism class of Γ (and vice-versa).

Why and how

Right-angled Coxeter groups are a rich source of examples in Geometric Group Theory.

We like to characterize properties of W by properties of Γ.
e.g.

- (Moussong) W is word hyperbolic if and only if each circuit in Γ of length four has a chord;
- Each separating subgraph of Γ corresponds to a splitting of W as a free product with amalgamation.

Tits '88

$$\operatorname{Aut} W = \operatorname{Aut}^0 W \rtimes \operatorname{Aut}^1 W$$

where

- $\operatorname{Aut}^0 W$ is the subgroup of basis-conjugating automorphisms (each generator is mapped to a conjugate of itself);
- $\operatorname{Aut}^1 W$ is a finite subgroup.

So

$$\mathcal{O}ut\, W \text{ is finite} \Leftrightarrow \operatorname{Aut}^0 W / \operatorname{Inn} W \text{ is finite.}$$

Laurence ’93 /Mühlherr ’98

For a vertex $v \in V$ and a connected component K of $\Gamma \setminus v^*$, the map

$$u \mapsto \begin{cases} vuv & \text{if } u \in K \\ u & \text{if } u \notin K \end{cases}$$

determines the partial conjugation $\chi_{vK} \in \operatorname{Aut}^0 W$.

$\operatorname{Aut}^0 W$ is generated by the set $\mathcal{P}$ of partial conjugations.

A partial conjugation

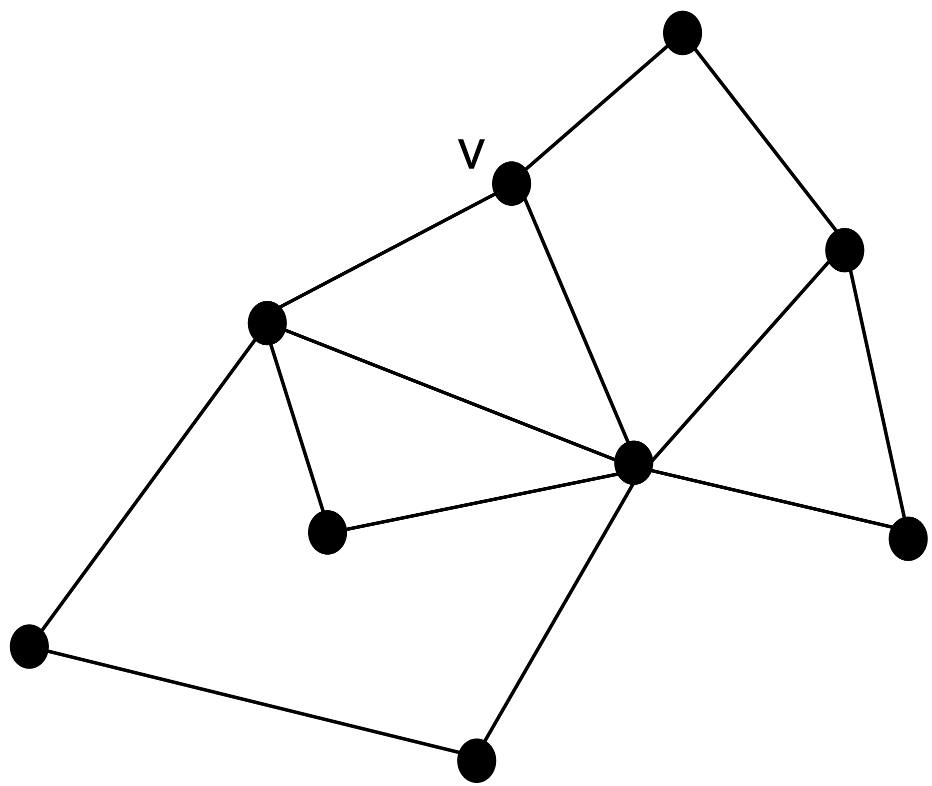


Figure 1: A graph Δ

A partial conjugation

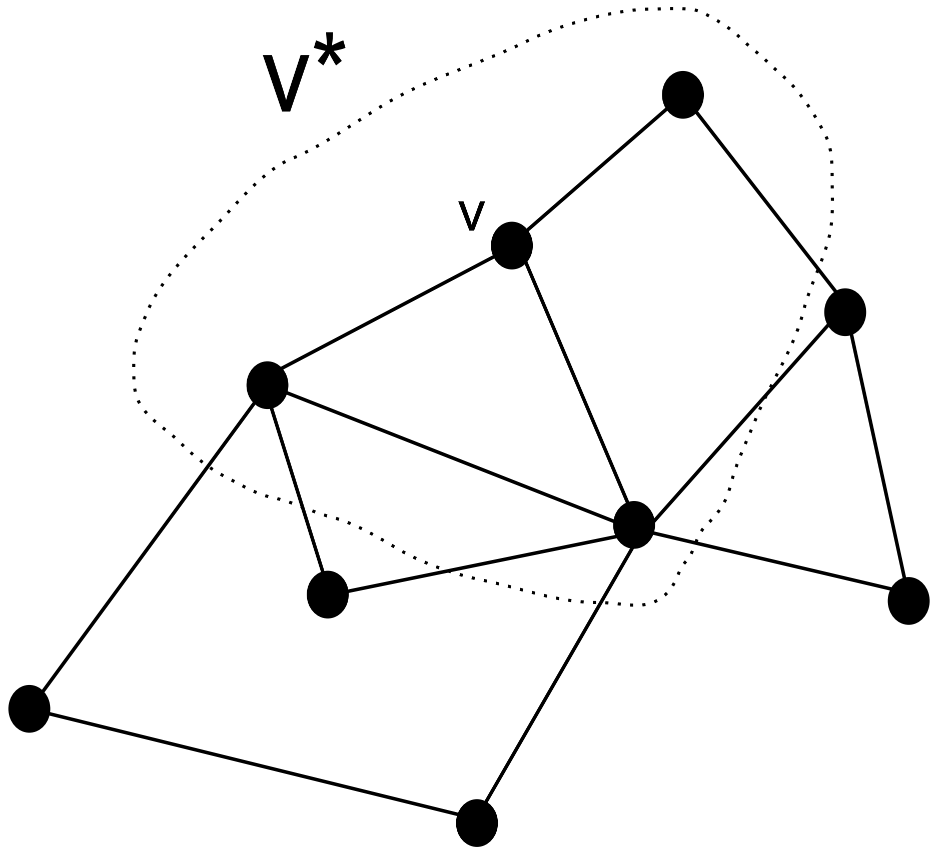


Figure 2: v^* in Δ

A partial conjugation

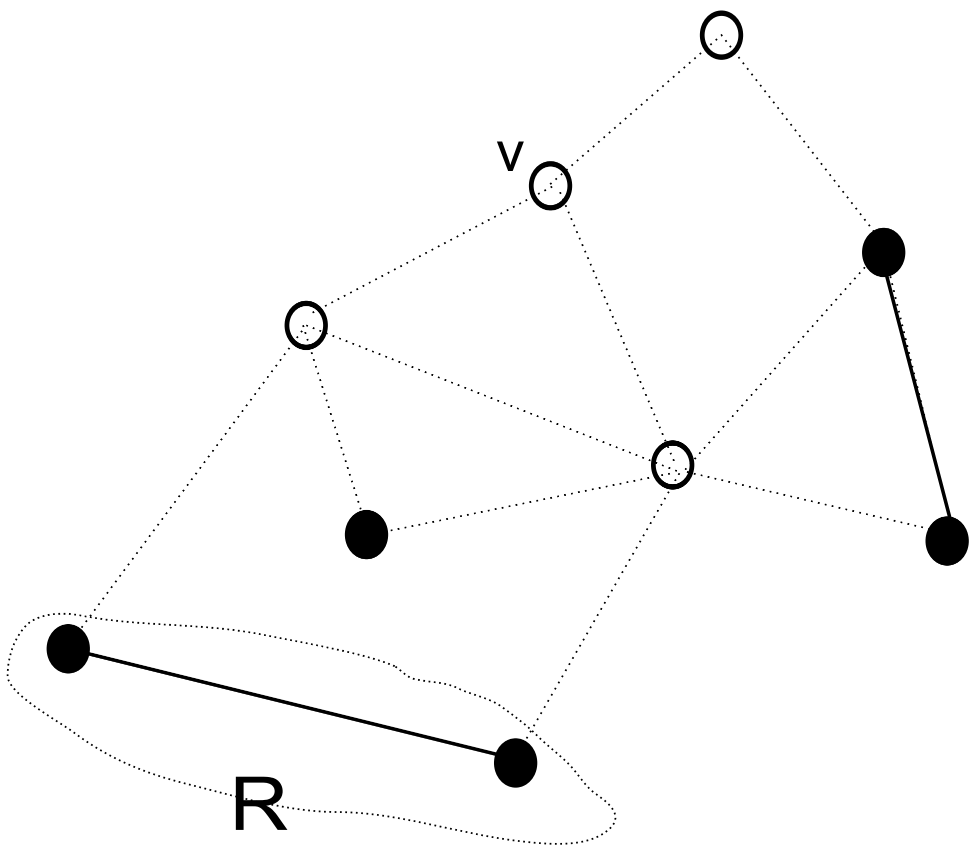


Figure 3: χ_{vR} is a partial conjugation

GPR '07

$$\mathrm{Aut}^0 W = \mathrm{Inn}\, W \rtimes \mathrm{Out}^0 W$$

where

- $\mathrm{Out}^0 W = \langle \mathcal{P}^0 \rangle$ for

  $\mathcal{P}^0 = \mathcal{P} \setminus \{\chi_{vK} \mid K$ the 'least' component of $\Gamma \setminus v^*\}$

So

$$\mathcal{O}ut\, W \text{ is finite} \Leftrightarrow \mathrm{Out}^0 W \text{ is finite.}$$

A graph property

We say that Γ contains a *separating intersection of links* (SIL) if there exist $u, v \in V$ such that:

1. $d(u,v) \geq 2$;
2. there exists a connected component R of $\Gamma \setminus (L_u \cap L_v)$ such that $u, v \notin R$.

$\mathcal{O}ut\, W$ is finite $\Leftrightarrow$ Γ does not contain a SIL.

A SIL

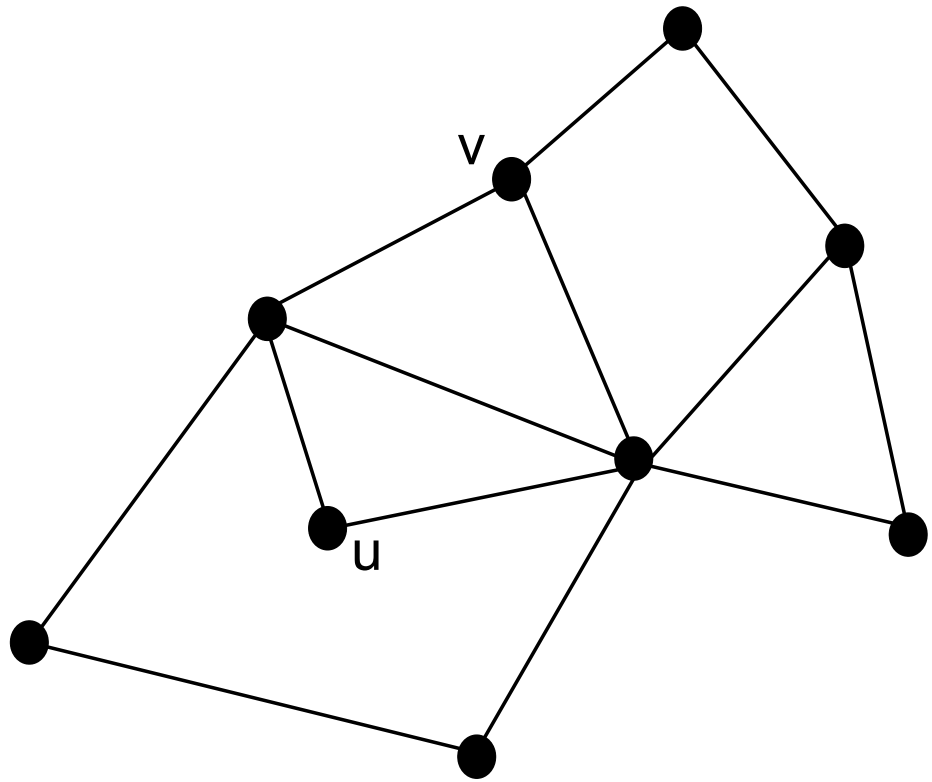


Figure 4: Δ

A SIL

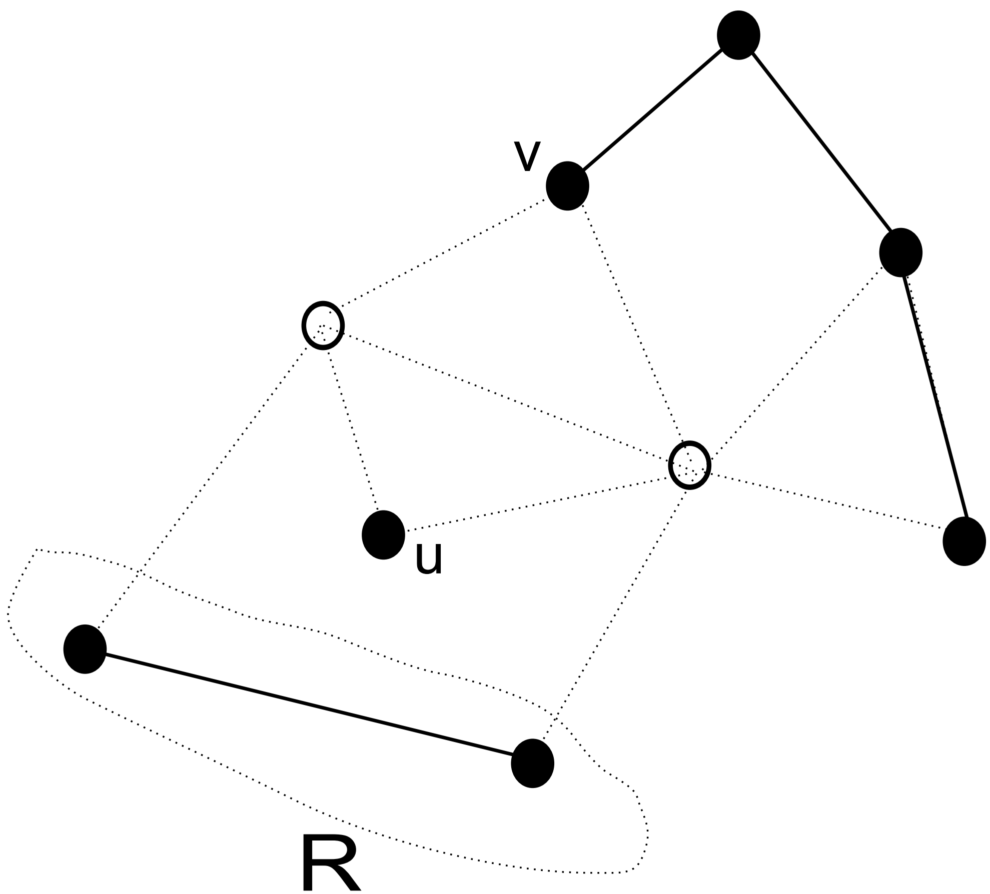


Figure 5: u, v and R make a SIL

A SIL

$$(\chi_{uR}\chi_{vR})^n(s) = \begin{cases} (uv)^n s (vu)^n & s \in R \\ s & s \notin R \end{cases}$$

$(\chi_{uR}\chi_{vR})^n \notin \operatorname{Inn} W$ for $n \neq 0$

Hence $\mathcal{O}ut\, W$ is infinite.

Not a SIL

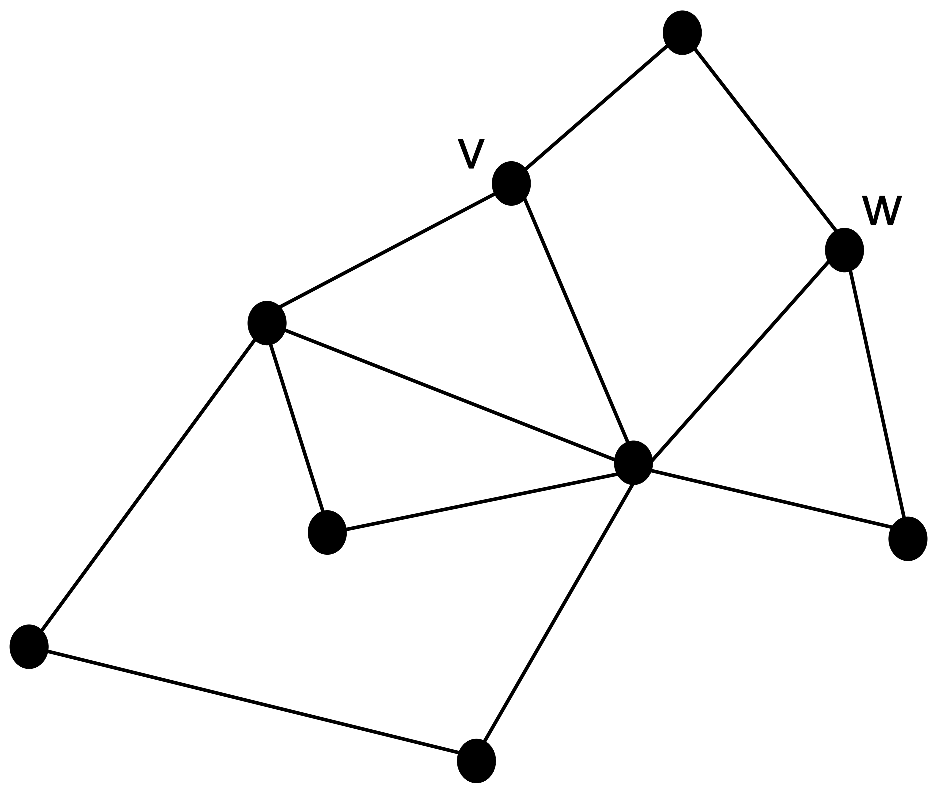
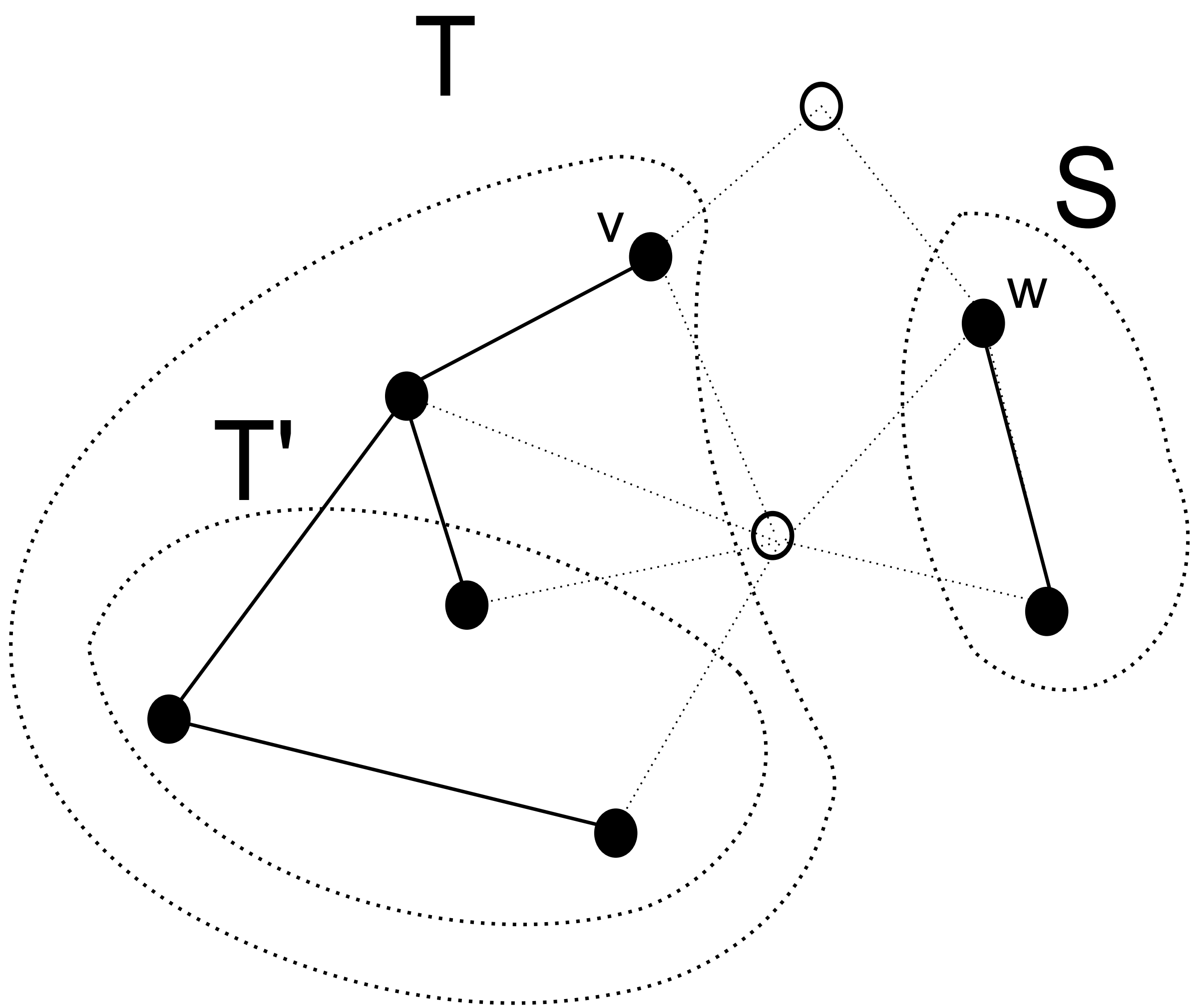


Figure 6: v and w are not part of a SIL

Not a SIL

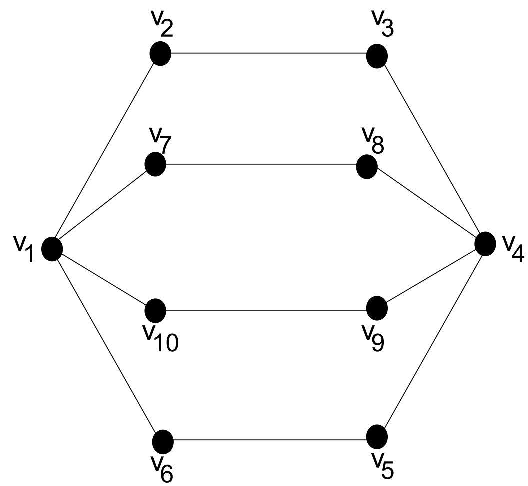


Not a SIL

$$\begin{aligned}
\chi_{vS}\chi_{wT}\chi_{vS}\chi_{wT} &= \iota_v\chi_{vT'}\chi_{wT}\iota_v\chi_{vT'}\chi_{wT} \\
&= \iota_v\chi_{vT'}\iota_w\iota_v\iota_w\chi_{wT}\chi_{vT'}\chi_{wT} \\
&= \iota_v\iota_w\iota_v\iota_w\chi_{vT'}\chi_{wT}\chi_{vT'}\chi_{wT} \\
&= \iota_v\iota_w\iota_v\iota_w \\
&\in \operatorname{Inn} W
\end{aligned}$$

(because $\chi_{vT'}$ and χ_{wT} commute).

Some applications



W is a one-ended word hyperbolic group with a non-trivial JSJ decomposition in the sense of Bowditch and $\mathcal{O}ut\, W$ is finite.

(cf. Miller-Neumann-Swarup '99)

Some applications

If W is one ended and word hyperbolic, then $\mathcal{O}ut\, W$ is finite
(cf. Levitt '05).

If $\mathcal{O}ut\, W$ is infinite and X is a CAT(0) space on which W acts geometrically, then the visual boundary ∂X is not locally connected
(uses Mihalik-Ruane-Tschantz '07).