Introduction to Fair Division

Problems that concern the fair division of resources are ubiquitous in society: Countries must share resources such as land, water, etc.; users of the internet share bandwidth; patients at a hospital share machinery and staff time; users of computers share servers; and so on. There is a branch of mathematics devoted to studying such problems. I will tell you a few things about it here.

Two comments first:

> 1) You won't learn anything here from which you will derive any serious practical benefit in the future. That is not my intent. I want to tell you about a field of mathematics that is interesting, and useful to the world. I simply want you to have some idea of how mathematics might come into this kind of problems. (Of course you might benefit someday from a hospital having a good method for allocating resources.)
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> 2) I will talk about sharing cakes. That makes the subject sound frivolous. Remember that the above examples are the ones that we really mean; the subject isn't truly about cutting cakes, that's just an amusing way of saying it.

Let's think about the simplest case: Two children, Abigail and Benjamin (A and B), want to share a cake. If the cake is one big homogeneous clump (imagine a clump of cheese cake), there is no interesting issue: To cut the clump into two halves of equal size is a matter of a steady hand. Mom or dad will be able to handle this just fine without any mathematics.

The problem becomes more interesting when the cake is composed of a strawberry half, and a cheese cake half. A likes strawberries better than cheese cake. B, by contrast, is not fussy at all, all that matters to him is volume. Now what should be done? In general, the central fact to keep in mind in fair division problems is this:

> *Different people value different components or aspects of the resource to be shared differently.*

You learned in kindergarten how to handle Abigail and Benjamin's problem: "I cut, you choose" is the answer. (Well, at least one answer. It is not the best answer by any means, as it turns out.) One of the children, let's say A, cuts the cake into two pieces that are of equal value to her. The other child, B, picks the piece that he likes better, and A gets the remaining piece. Of course, A will try hard to generate two pieces that are of equal value to her, since she does not know in advance which of the two will become hers.

"I cut, you choose" guarantees that each of the two children will receive a piece that they consider worth at least half the cake. (Notice that A does not go by volume. A piece that is less than half the volume of the cake may still be considered worth more than half the cake by A.) We therefore say that "I cut, you choose" generates a *fair* division.

DEFINITION 1: A division of a cake among N people is *fair* if each person thinks that his or her share is at least $1/N$-th of the cake, in value.

For each of the N people ($N = 2$ in our example, the two people being A and B), a *fair share* is a share that he or she considers worth at least $1/N$-th of the cake.

"I cut, you choose" is a way of producing fair shares for everybody, and therefore a fair division, when $N = 2$. It was described in the Bible:

> *And Abram said unto Lot, let there be no strife, I pray thee, between me and thee, and between my herdmen and thy herdmen; for we be brethren. Is not the whole land before thee? Separate thyself, I pray thee, from me: if thou wilt take the left hand, then I will go to the right; or if thou depart to the right hand, then I will go to the left.* (Genesis 13:8-9, King James Version)

It took until 1943 before mathematicians realized that there is an interesting problem to think about here: What if $N > 2$? The first mathematicians who became interested in this question were members of a distinguished group of mathematics professors at the University of Lwów: Hugo Steinhaus, Stefan Banach, and Bronislaw Knaster. At the time, Lwów was under German occupation, the university had been closed, Steinhaus was in hiding (he was Jewish), and Banach was employed as a lice feeder in a typhus research laboratoy. I don't know what was Knaster's fate during that time. (All three survived the Nazi occupation and the war.)

Here is Steinhaus' solution to the problem of three children (A, B, and C) sharing a cake. First, A divides the cake into three pieces that are of equal value in her eyes. Then B and C each declare which of those three pieces they consider fair shares. If the total number of pieces named by at least one of them is greater than one, there is no problem: We assign to each of them a piece that they consider a fair share, then give to A the remaining piece — which she considers a fair share, since she cut the cake in such a way that each of the three pieces is a fair share to her. But what if B and C each declare that there is only one piece among the three pieces cut by A that they consider a fair share? In this case, we give A one of the other two pieces (less than one third of the cake, in B and C's opinion), re-combine the remaining too pieces (more than two thirds of the cake, in B and C's opinion), and let B and C play "I cut, you choose" over them.

This is called *Steinhaus' lone divider method*. ("Lone divider" because one person cuts, the others choose.) You will be able to see right away:

THEOREM 1: Steinhaus' lone divider method guarantees that each of the three participants will get a fair share.

How about $N > 3$? It turns out that Steinhaus' idea has a generalization to any number N of participants, but the generalization is not quite so simple. It was first described by Harold Kuhn, a game theorist at Princeton University, in the 1960s, and is now known as the *Steinhaus-Kuhn method*. I cannot describe it in the time I have here.

Some fair divisions are better than others. For instance, some create *envy* while others don't. If A, B, and C share a cake, A might, in his opinion, get one third of the cake, but see B walk away with a piece that is worth, in A's opinion, one half of the cake. So A would admit that he got his fair share, yet he would envy B.

DEFINITION 2: A division of a cake among N people is *envy-free* if each participant thinks that nobody else gets a share more valuable than his or her own share.

When $N = 2$, envy-freeness and fairness are the same thing, but when $N > 2$, envy-freeness is a more stringent condition than fairness (see Exercise 1).

THEOREM 2: Steinhaus' lone divider method does not guarantee an envy-free division.

PROOF: All we need to do is to produce a single example in which A will envy B, and that is easy. Suppose that the cake consists of strawberry, rhubarb, and chocolate components of equal volume. A is not fussy: She is only interested in volume. She cuts the cake into its strawberry, rhubarb, and chocolate components — each component has the same volume, and therefore the same value to her. However, B and C don't eat rhubarb (they don't know what it is), and are allergic to strawberries. Therefore, each of them considers the chocolate component the only fair share. (In fact, each of them considers the chocolate component one hundred percent of the cake!) Therefore A is assigned one of the two non-chocolate components at random — say the rhubarb component. B and C then re-combine the strawberry and chocolate components, and play "I cut, you choose".

Let's say that B cuts the re-combined strawberry and chocolate components into two pieces, equal in value in his eyes. He does not care about the strawberry component at all, so he makes one piece equal to half the chocolate component and the entire strawberry component, the other piece equal to the other half of the chocolate component. (This is considerate of him, by the way. He does this just in case C wants the strawberry component; he doesn't know that C does not want that component either.) C does not care about the strawberry component, so he takes one half of the chocolate component, and leaves the other half of the chocolate component, plus the entire strawberry component, to B.

Now A will envy B! QED

There is a variation on Steinhaus' lone divider method that does guarantee envy-freeness. It is called the *Selfridge-Conway* method, proposed independently by the mathematicians John Selfridge and John Conway. I don't have time to describe it here.

There are other requirements that one might want to impose on cake divisions. For instance:

DEFINITION 3: A division of a cake among N people is *optimal* if there is no alternative division that is better for at least one of the N people, without being worse for any of the others.

THEOREM 3: “I cut, you choose” does not guarantee an optimal division.

PROOF: Suppose John and Jane share a cake that is half chocolate, half cherry. John likes any cake, he goes by volume. Jane does not care too much for cherries. John does not know that about her. (They have only been married for eight years.) Therefore, when John cuts the cake, he divides it into two equal halves, cutting both the chocolate and the cherry components down the middle. Jane takes one of the two halves — since they are identical, it does not matter to her which one — and John takes the other.

This is a fair division (each of the two gets half the cake — by volume, by value, and by any other measure), but it is not optimal: A division in which John would get the cherry half, and Jane the chocolate half would be just as good for John, but better for Jane. QED

Steinhaus’ lone divider method does not guarantee optimality either (Exercise 4), and neither does the envy-free method of Selfridge and Conway.

DEFINITION 4: A division of a cake among N people is *equitable* if each of the N people gives the same answer to the question “Which fraction of the value of the cake does your share represent?”

You may agree that equitability would be nice when you divide a cake.

THEOREM 4: “I cut, you choose” does not guarantee an equitable division.

PROOF: The cutter is guaranteed half the cake (in his or her eyes). The chooser may well get more than half the cake, by value. QED

So Abram was generous when he offered to cut and let Lot choose. Steinhaus’ lone divider method does not guarantee an equitable division either (Exercise 5), and neither does the envy-free method of Selfridge and Conway.

This has been a rather pessimistic lecture. All the methods I told you about have serious flaws: None is optimal, none is equitable, and Steinhaus' method is not even envy-free. But of course, that's an opportunity for mathematicians: They can try to figure out how to do fair division better. I will end by very briefly sketching some more recent, and more positive results in this area.

When the cake consists of finitely many homogeneous (objectively divisible) clumps, there is exactly one method that guarantees envy-freeness, optimality, and equitability for $N = 2$. It is called the *adjusted winner method*, and was proposed by Stephen Brams (a political scientist at NYU) and Alan Taylor (a mathematician at Union College) in the 1990s. NYU holds a patent on this method! (So be careful next time you cut a birthday cake, you might owe royalties.)

Again assuming that the cake consists of finitely many homogeneous (objectively divisible) clumps, there provably is no method combining the three properties of envy-freeness, optimality, and equitability for $N > 2$. However, any two of the three can be obtained simultaneously even when N is greater than two.

Exercises

1. Explain: a) When $N = 2$, envy-freeness and fairness are the same thing. b) When $N > 2$, envy-freeness implies fairness, but fairness does not imply envy-freeness.

2. Explain the following facts about Steinhaus' lone divider method: a) A, the divider, may envy B or C, but not both. b) B and C will never envy anybody.

3. In the example in the proof of Theorem 2, if you asked A, B, and C in the end "Which percentage of the cake do you think you got?", what would the answers be?

4. Explain: The division in the proof of Theorem 2 is not optimal.

5. Explain: The division in the proof of Theorem 2 is not equitable.

6. Assume that the cake consists of finitely many homogeneous (objectively divisible) clumps. Propose a simple division method that guarantees, for any N, an envy-free and

equitable division.[1] (By necessity, such a method cannot guarantee optimality as well when $N > 2$.)

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[1] As stated in the last paragraph of the text, any two of the three properties of envy-freeness, optimality, and equitability can be guaranteed by a fair division method for any N. The pair envy-freeness + equitability is the subject of this exercise, and it is simple. The other two pairs, envy-freeness + optimality and optimality + equitability, are not so simple.