

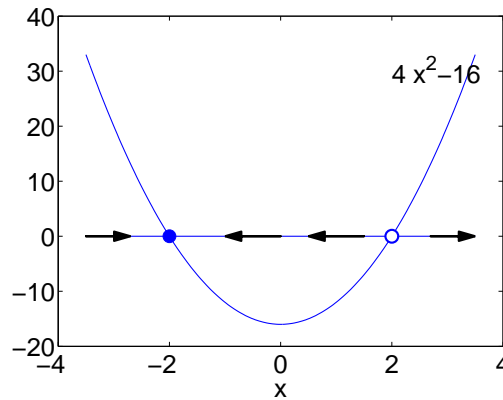
Solutions for Homework 1

1. 2.1.1 Fixed points are points where $\sin x = 0$, that is, integer multiples of π .

2.1.2 The greatest velocity to the right occurs where $\sin x$ is the greatest. That is, where x is $\pi/2$ plus an integer multiple of 2π .

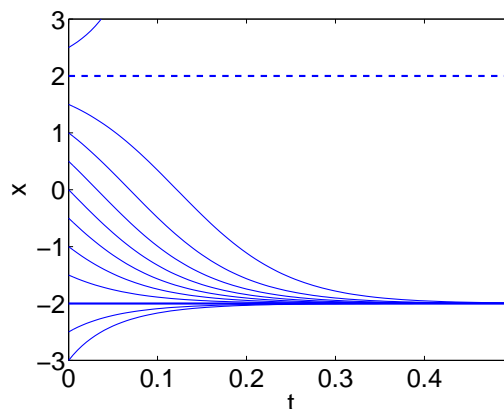
2.1.3 a) $\dot{x} = \sin x$ implies $\ddot{x} = (\cos x)\dot{x}$ by the chain rule, so $\ddot{x} = \cos x \sin x = \sin(2x)/2$. Did you remember that $\cos x \sin x = \sin(2x)/2$? If not, here is an easy way to memorize it: Among the trigonometric formulas, the only ones that I memorize are $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$. All others follow from these very quickly. For example, the first with $a = b = x$ gives you what you need for this problem. b) The acceleration to the right is greatest where $\sin(2x)$ is greatest, that is where $x = \pi/4$ plus an integer multiple of π .

2. 2.2.1. Sketch of the vector field:



$x = -2$ is a stable fixed point, and $x = 2$ is unstable, as is apparent from the vector field.

Here is what the solutions look like:



(Solutions with $x > 2$ blow up in finite time.)

Solution in closed form: If $x_0 = \pm 2$, then $x(t) = \pm 2$ for all t , since 2 and -2 are fixed points. Let's assume that $x_0 \neq \pm 2$ now. Then $x(t) \neq \pm 2$ for all t (this is true by the uniqueness part of

the existence and uniqueness theorem). This will be used in the following manipulations.

$$\begin{aligned} \dot{x} = 4x^2 - 16 &\Leftrightarrow \frac{\dot{x}}{x^2 - 4} = 4 \Leftrightarrow \frac{4\dot{x}}{(x-2)(x+2)} = 16 \Leftrightarrow \\ \frac{\dot{x}}{x-2} - \frac{\dot{x}}{x+2} = 16 &\Leftrightarrow \frac{d}{dt} \ln(|x-2|) - \frac{d}{dt} \ln(|x+2|) = 16 \Leftrightarrow \\ \frac{d}{dt} \ln\left(\left|\frac{x-2}{x+2}\right|\right) = 16 &\Leftrightarrow \ln\left(\left|\frac{x-2}{x+2}\right|\right) = 16t + C \end{aligned}$$

for some constant C , which is found by plugging in $t = 0$. Using the notation $x(0) = x_0$, we find

$$C = \ln\left(\left|\frac{x_0 - 2}{x_0 + 2}\right|\right).$$

So

$$\ln\left(\left|\frac{x-2}{x+2}\right|\right) = 16t + \ln\left(\left|\frac{x_0 - 2}{x_0 + 2}\right|\right).$$

Exponentiate both sides of this equation:

$$\left|\frac{x-2}{x+2}\right| = \left|\frac{x_0 - 2}{x_0 + 2}\right| e^{16t}.$$

So

$$\frac{x-2}{x+2} = \pm \frac{x_0 - 2}{x_0 + 2} e^{16t}.$$

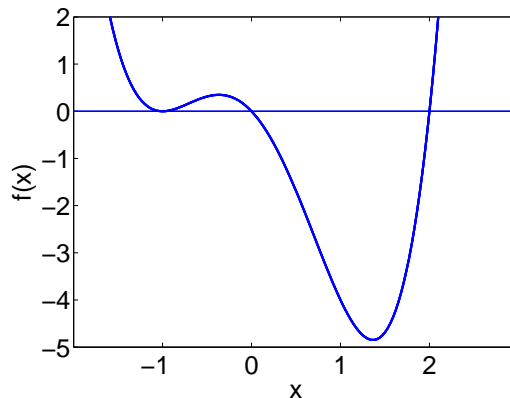
For $t = 0$, the correct sign is $+$. Therefore the correct sign must be $+$ for all t . (If the sign changed from $+$ to $-$ all of the sudden, x would have to be discontinuous at that point!) So

$$\frac{x-2}{x+2} = \frac{x_0 - 2}{x_0 + 2} e^{16t}.$$

Multiply this equation by $x + 2$, then solve for x :

$$x(t) = 2 \frac{(x_0 + 2) + (x_0 - 2)e^{16t}}{(x_0 + 2) - (x_0 - 2)e^{16t}}$$

3. 2.2.8 It is best to draw a suitable function first:



After that, it's almost as easy to write down a formula defining a function that looks like that, qualitatively:

$$f(x) = x(x+1)^2(x-2)$$

(In fact, the above plot shows precisely that function.)

2.2.9 The function must vanish (be zero) at $x = 0$ and $x = 1$. It must be negative between 0 and 1, and positive everywhere else. Its minimum must occur near $x = 1/2$. It must increase for $x > 1$, and decrease for $x < 0$. (Can you see why?) A function that meets all these criteria is

$$f(x) = x(x-1) .$$

4. 2.3.4 To motivate this problem, suppose that in general

$$\dot{N} = f(N)$$

describes population growth. We certainly want

$$f(0) = 0.$$

This just expresses the fact that it is possible that there is no population at all, and never will be.

Now what are the simplest choices of functions $f = f(N)$ with $f(0) = 0$? The very simplest one is linear:

$$f(N) = rN.$$

This gives rise to exponential growth (or, if $r < 0$, decay). The second-simplest is quadratic:

$$f(N) = r \left(1 - \frac{N}{N_\infty} \right) N.$$

This gives rise to logistic growth, as we discussed in class; N_∞ is the carrying capacity. (The physically interesting case is $r > 0$. What happens if $r < 0$?) This problem is about the third-simplest case, that of a cubic function f . Since $f(0)$ is to be zero, we must be able to write $f(N)$ as N times a quadratic function. Strogatz writes it like this:

$$f(N) = (r - a(N - b)^2) N \tag{1}$$

Any quadratic function can be written in the form $r - a(N - b)^2$. So our population growth equation is then.

$$\dot{N} = (r - a(N - b)^2) N. \tag{2}$$

Let us first consider the case when $N = 0$ is the *only* fixed point, that is, when the equation

$$r - a(N - b)^2 = 0 \tag{3}$$

has no solution. One case in which Eq. (3) has no solution is $a = 0, r \neq 0$. In that case, Eq. (2) becomes

$$\dot{N} = rN,$$

the exponential growth (or, if $r < 0$, decay) model. This is not of interest to us here, so let us assume $a \neq 0$. In that case, Eq. (3) has no solution if and only if

$$\frac{r}{a} < 0. \quad (4)$$

There are two possibilities: Either $N = 0$ is a stable fixed point, or an unstable one. Which of these two cases we are in depends on the sign of $f'(0)$. In general,

$$f'(N) = r - a(N - b)^2 - 2a(N - b)N,$$

and therefore

$$f'(0) = r - ab^2.$$

So $N = 0$ is stable if $r < ab^2$, and unstable if $r > ab^2$. Remember that we assume (4), that is, we assume that r and a are of opposite signs. Therefore if $a > 0$, then certainly $r < ab^2$, since then r is negative and ab^2 is positive. On the other hand, if $a < 0$, then certainly $r > ab^2$, since then r is positive and ab^2 is negative. This argument shows that in fact, if (3) has no solution, then $N = 0$ is a stable fixed point if $a > 0$, and an unstable one if $a < 0$. We could have seen this by a different argument as well: For large N ,

$$f(N) \sim -aN^3$$

(compare Eq. (1)). Therefore, if $a < 0$, then $f(N) \rightarrow -\infty$ as $N \rightarrow -\infty$, and $f(N) \rightarrow +\infty$ as $N \rightarrow +\infty$. If f crosses through $N = 0$ only once, it must cross from negative values (for $N < 0$) to positive ones (for $N > 0$), and therefore $N = 0$ must be unstable. One sees by a precisely analogous argument that $N = 0$ must be a stable fixed point if Eq. (3) has no solution, and $a > 0$.

But if $N = 0$ is the only fixed point, and is stable, the problem is uninteresting: The population simply goes extinct, no matter where we start. If $N = 0$ is the only fixed point, and it is unstable, then in fact Eq. (2) is unphysical, since it predicts blowup of the population size in finite time. (Do you see why?)

So we have now concluded that the case in which Eq. (3) has no solution is uninteresting. We will now assume

$$\frac{r}{a} > 0, \quad (5)$$

so that Eq. (3) has two solutions. Those two solutions are

$$N_{\pm} = b \pm \sqrt{\frac{r}{a}}. \quad (6)$$

Either one of the two solutions of Eq. (3) could be zero, depending on the parameter values, but we will assume that they are not zero, so that there are three distinct solutions of $f(N) = 0$, rather than a solution of multiplicity two at $N = 0$.

There are three cases to distinguish now:

$$N_- < N_+ < 0, \quad (7)$$

$$N_- < 0 < N_+, \quad (8)$$

and

$$0 < N_- < N_+. \quad (9)$$

The negative fixed points are of no physical interest to us, so the number of physically interesting fixed points is one in case (7), two in case (8), and three in case (9). Case (7) either leads to guaranteed extinction of the population, if $a > 0$, or blowup in finite time, if $a < 0$; the former is uninteresting, the latter unphysical.

So let us proceed to case (8). Here there are again two possibilities. If $a > 0$, the fixed point 0 is unstable, and the fixed point N_+ is stable. Therefore the population will level off at N_+ for any initial value $N(0) > 0$. This is very similar to logistic growth, and therefore not so interesting to us here. The other possibility is $a < 0$, in which case the population will go extinct if $N(0) < N_+$, and blow up in finite time if $N(0) > N_+$ — not a physical case.

So let us proceed to case (9). Again, we think about the two possibilities $a > 0$ and $a < 0$. If $a < 0$, then the population will level off to N_- if $0 < N(0) < N_+$, and blow up in finite time if $N(0) > N_+$ — not a physical case. The only potentially interesting and physical case is therefore that in which (9) holds and $a > 0$. In this case, the population will go extinct if $0 < N(0) < N_-$, and level off to N_+ if $N(0) > N_-$. Thus N_- is an extinction threshold: If $N(0)$ is below it, the population goes extinct, if $N(0)$ is above it, the population levels off to the “carrying capacity” N_+ .

I summarize: The only choice of a cubic f which is interesting, physically plausible, and qualitatively different from exponential and logistic growth is that in which

$$a > 0, \quad r > 0, \quad b > \sqrt{\frac{r}{a}}.$$

In this case, there are three fixed points, namely

$$0, \quad N_- = b - \sqrt{\frac{r}{a}}, \quad N_+ = b + \sqrt{\frac{r}{a}},$$

and

$$0 < N_- < N_+.$$

N_- is the extinction threshold, and N_+ the carrying capacity.