Solutions for Homework 2

1. 3.1.1 The graph of

$$f(x) = 1 + rx + x^2$$

is a parabola, opening upwards. The minimum occurs where $f'(x) = 0$, that is, at

$$x_0 = -\frac{r}{2}.$$

At this point, the value of f is

$$f(x_0) = 1 - \frac{r^2}{2} + \frac{r^2}{4} = 1 - \frac{r^2}{4}.$$

If $|r| < 2$, then $f(x_0) > 0$, therefore $f(x) > 0$ for all x, and there are no fixed points. If $|r| > 2$, then $f(x_0) < 0$, and therefore there are two fixed points. They are easy to compute:

$$f(x) = 0 \quad \Leftrightarrow \quad x = \frac{-r \pm \sqrt{r^2 - 4}}{2}\,.$$

We will write

$$x_\pm = \frac{-r \pm \sqrt{r^2 - 4}}{2}\,. \tag{1}$$

Since the graph of f is a parabola opening upwards, it must be true that $f'(x_-) < 0$ and $f'(x_+) > 0$, therefore x_- is stable and x_+ is unstable. Let us plot the bifurcation diagram. This requires plotting x_- and x_+ as functions of r. We could do that, but life becomes easier if we solve the equation

$$1 + rx_\pm + x_\pm^2 = 0$$

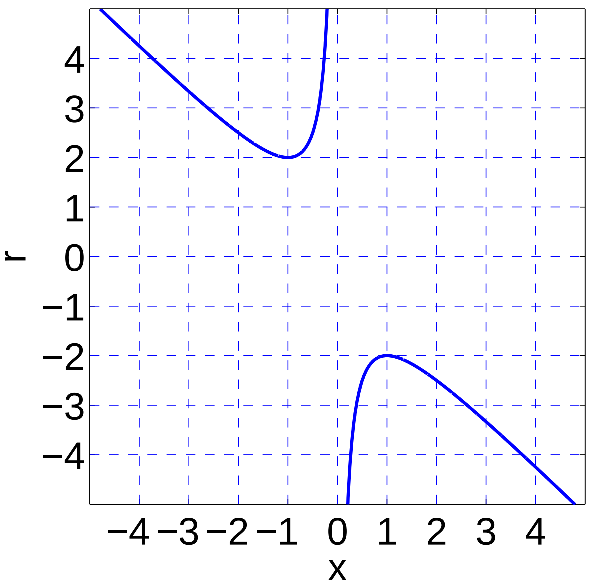
for r:

$$r = -x_\pm - \frac{1}{x_\pm}\,.$$

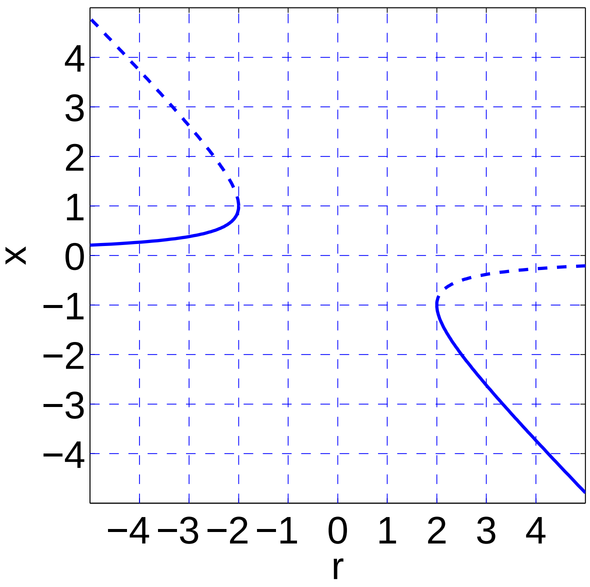
The function

$$r(x) = -x - \frac{1}{x}$$

is really easy to plot:



Now the graph of x as a function of r is obtained simply by reversing x and r in the above plot. Here is what you get when you do that, remembering that the smaller of the two fixed points is stable, and the larger is unstable:



This is the bifurcation diagram. There are two saddle-node bifurcations, one at $r = -2$ and the other at $r = 2$. I'll leave it to you to plot the qualitatively different vector fields.

3.1.3 Let's first think about the fixed point:

$$f(x) = r + x - \ln(1+x).$$

This function is defined for $x > -1$. As $x \to -1$, $f(x) \to \infty$. It may not be immediately clear to you what happens as $x \to \infty$: x goes to infinity, but $-\ln(1+x)$ to $-\infty$. Which term wins? The answer is that x wins. For large x, $\ln(1+x)$ is much smaller than x. You can see this for instance by proving, using l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = 0.$$

So $f(x)$ tends to infinity as $x \to -1$ or $x \to \infty$. What happens in between? Compute the derivative:

$$f'(x) = 1 - \frac{1}{1+x}.$$

Set it equal to zero: You find that there is only one place where $f'(x) = 0$, namely $x = 0$. Therefore the absolute minimum of x must occur at $x = 0$. The minimum value is

$$f(0) = r.$$

When $r > 0$, then $f(x) > 0$ for all x, so there is no fixed point. When $r < 0$, then $f(x) < 0$, so there are two fixed points. A saddle-node bifurcation occurs as r passes through 0.

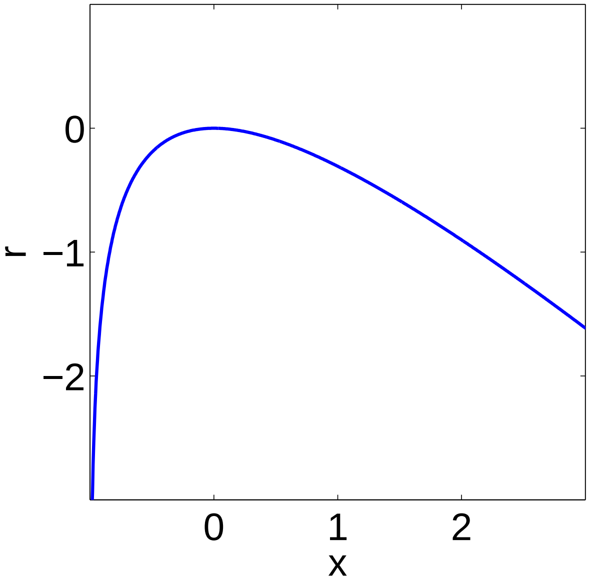
Let us now think about the case when $r < 0$, and denote the two fixed points by x_- and x_+, with the convention that $x_- < x_+$. It is clear then that x_- is stable, and x_+ is unstable. To plot the bifurcation diagram, we should express x_- and x_+ as functions of r, and it is unclear how to do that. However, the same trick that we used for problem 3.1.1 works here as well: Don't solve for x, solve for r:

$$r + x - \ln(1+x) = 0 \Leftrightarrow r = \ln(1+x) - x.$$

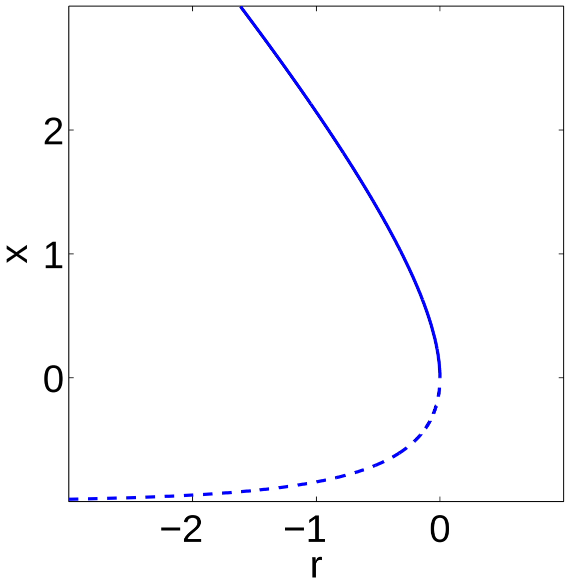
The function

$$r(x) = \ln(1+x) - x$$

is easy to plot: As $x \to -1$, it tends to $-\infty$. For large x, $r(x) \sim -x$ tends to $-\infty$ as well. The maximum occurs at $x = 0$, and the maximum value of r is $r = 0$. So here is what it looks like:



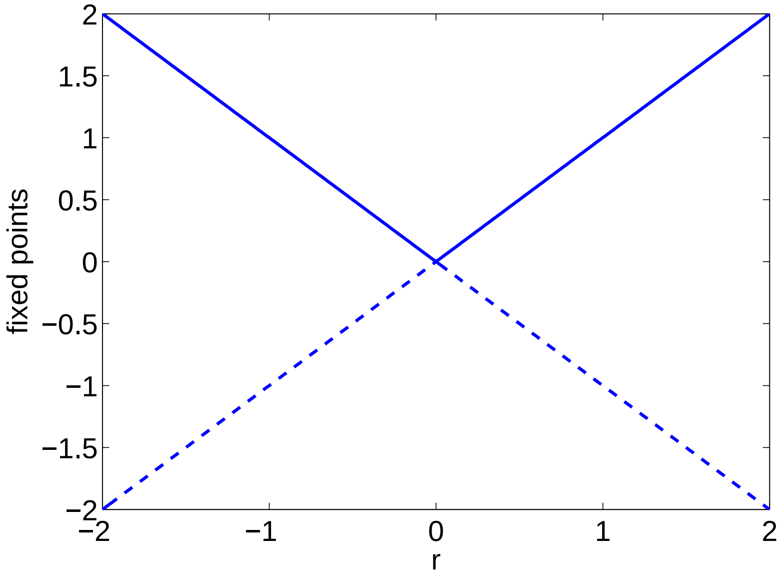
To plot the fixed points as a function of r, flip the coordinate axes, and remember that the smaller of the two fixed points is stable:



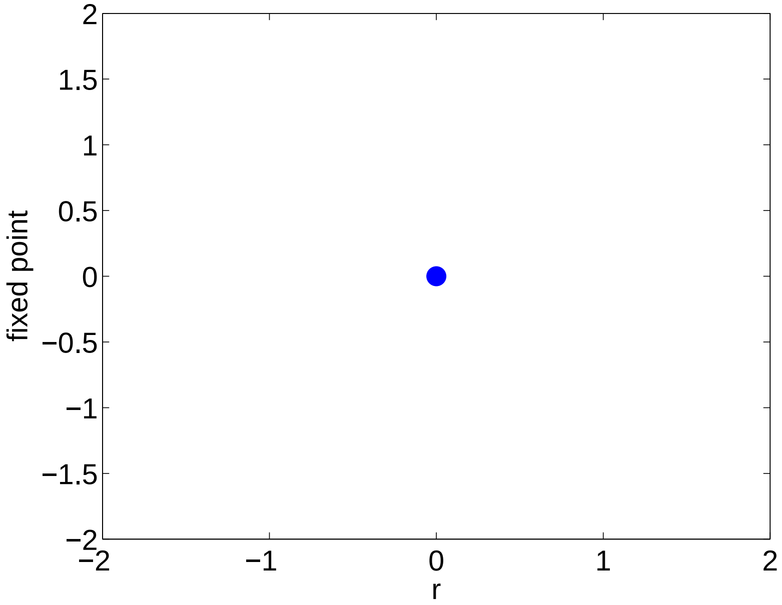
2. 3.1.5 a) For any $r \neq 0$, there are two fixed points, one at $-|r|$ and the other at $|r|$. The function

$$f(x) = r^2 - x^2$$

is a parabola that opens downwards, and therefore the fixed point $|r|$ is stable, the fixed point $-|r|$ is unstable. When $r = 0$, there is a single semi-stable fixed point at 0. The bifurcation diagram looks like this:



b) There are no fixed points, except for $r = 0$, when there is a semi-stable fixed point at $x = 0$. For $r \neq 0$, the flow is always from left to right. The "bifurcation diagram", if one should call it that, looks like this:



4. 3.3.1 The differential equation is

$$\dot{n} = Gn\frac{p}{Gn + f} - kn. \tag{2}$$

Pause to think about how this differs from the laser model of Section 3.3, which we discussed in class. In Section 3.3, we assumed that the number N of excited atoms is

$$N = N_0 - \alpha n, \tag{3}$$

This can only be true if n is not too large — otherwise N, the number of excited atoms, gets negative! Here we write instead

$$N = \frac{p}{Gn + f}. \tag{4}$$

Now N cannot get negative, and tends to 0 as $n \to \infty$, as it should. We can relate (4) to (3) by defining

$$N_0 = \frac{p}{f}.$$

With this notation, (4) becomes

$$N = \frac{N_0 f}{Gn + F} = \frac{N_0}{(G/f)n + 1}.$$

We will now use the fact that for small $|\varepsilon|$,

$$\frac{1}{1+\varepsilon} \approx 1 - \varepsilon$$

(this is the linear approximation of $1/(1+\varepsilon)$ at $\varepsilon = 0$). This implies that for small $(G/f)n$,

$$N = \frac{N_0}{(G/f)n+1} \approx N_0\,(1-(G/f)n)\,.$$

If we write

$$\alpha = \frac{G}{f}N_0 = \frac{G}{f}\frac{p}{f} = \frac{Gp}{f^2},$$

this becomes

$$N \approx N_0 - \alpha n.$$

So in summary, (4) has the advantage over (3) that it does not allow negative N, but as long as

$$n \ll \frac{f}{G},$$

(4) is approximately the same as (3), with

$$N_0 = \frac{p}{f} \quad \text{and} \quad \alpha = \frac{Gp}{f^2}\,.$$

Back to Eq. (2). Clearly $n = 0$ is a fixed point. There is a second fixed point, obtained by solving the equation

$$\frac{Gp}{Gn+f} - k = 0.$$

This equation always has a solution:

$$\begin{aligned}
\frac{Gp}{Gn+f} &= k \;\Leftrightarrow \\
\frac{Gn+f}{Gp} &= \frac{1}{k} \;\Leftrightarrow \\
Gn+f &= \frac{Gp}{k} \;\Leftrightarrow \qquad (5)\\
n &= \frac{p}{k} - \frac{f}{G}
\end{aligned}$$

We write

$$n^* = \frac{p}{k} - \frac{f}{G}\,.$$

We note, from (5):

$$Gn^* + f = \frac{Gp}{k}. \qquad (6)$$

This will be used shortly.

Now let's think about stability of the fixed points. The right-hand side of Eq. (2) is

$$F(n) = Gn\frac{p}{Gn+f} - kn.$$

(I write $F(n)$, not $f(n)$, because the letter "f" denotes a parameter in this problem.) The derivative of F is

$$F'(n) = \frac{pG}{Gn+f} - n\frac{pG^2}{(Gn+f)^2} - k.$$

So

$$F'(0) = \frac{pG}{f} - k,$$

and, using Eq. (6),

$$F'(n^*) = \frac{pG}{Gp/k} - \left(\frac{p}{k} - \frac{f}{G}\right)\frac{pG^2}{(Gp/k)^2} - k = -\left(\frac{p}{k} - \frac{f}{G}\right)\frac{k^2}{p}.$$

So if

$$\frac{pG}{f} - k < 0, \qquad (7)$$

then 0 is a stable fixed point, and n^* is unstable. If on the other hand

$$\frac{pG}{f} - k > 0, \qquad (8)$$

then 0 is unstable, and n^* is stable. Conditions (7) and (8) can be written as $p < p_c$ and $p > p_c$, respectively, with

$$p_c = \frac{kf}{G}.$$

As p exceeds the critical pump strength p_c, the fixed point 0 becomes unstable, and n^* becomes stable. Note that for $p = p_c$, the two fixed points coincide: $n^* = 0$ when $p = p_c$. This is a transcritical bifurcation — the two fixed points "collide and exchange their stability properties", as it were.