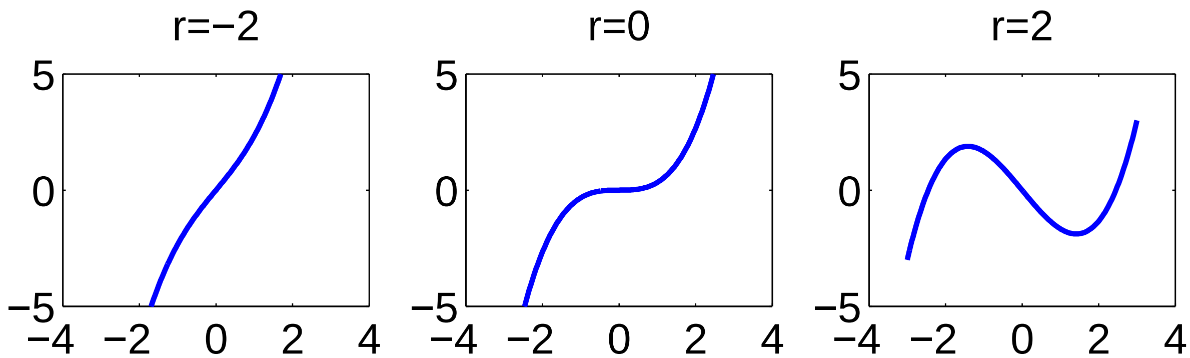
Solutions for Homework 3

1. 3.4.16 a) The potential is

$$V(x) = -rx + \frac{x^3}{3}.$$

You could add a constant C, but that would not make any essential difference; see below. Here is the graph of V for $r = -2$, $r = 0$, and $r = 2$:



The fixed points of the differential equation are the stationary points of V, that is, the values x for which $V'(x) = 0$. Local minima are stable fixed points, since in a local minimum, $V'(x)$ changes from negative to positive, and therefore $-V'(x)$ changes from positive to negative. Local maxima are unstable fixed points. Stationary points that are neither local maxima, nor local minima are semi-stable points.

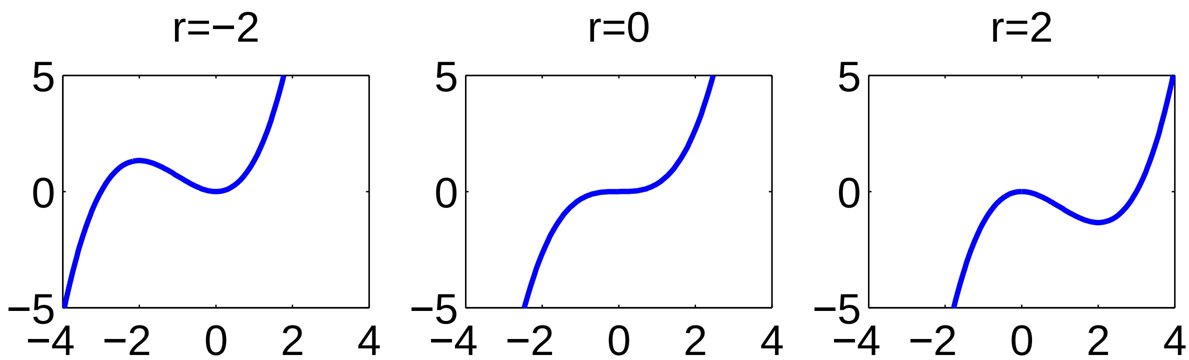
The potential V is defined to be the *negative* of an anti-derivative of the right-hand side of the differential equation. Why the minus sign? The answer is this: If we were to leave out the minus sign, simply writing the equation in the form $\dot{x} = dV/dx$, then stable fixed points would be local maxima of V, and unstable ones would be local minima. This is unintuitive, if you think of a little ball rolling on the graph of V!

Back to our example. There are no stationary points of V for $r < 0$, exactly one for $r = 0$ (namely, 0), and two for $r > 0$ (namely, $\pm\sqrt{r}$). If we had added a constant C to V, the graph of V would have shifted upwards or downwards, but the stationary points would have remained the same.

b) The potential is

$$V(x) = -r\frac{x^2}{2} + \frac{x^3}{3}.$$

Here is the graph of V for $r = -2$, $r = 0$, and $r = 2$:



For any $r \neq 0$, there are two stationary points of V, and therefore two fixed points of the differential equation, namely $x = 0$ and $x = r$. For $r = 0$, there is only one, namely $x = 0$. For $r < 0$, $x = r$ is a local maximum of V, and therefore an unstable fixed point of the differential equation, and $x = 0$ is a local minimum of V, and therefore a stable fixed point. For $r = 0$, $x = 0$ is a stationary point, but neither a local maximum nor a local minimum of V, an therefore $x = 0$ is a semi-stable fixed point. For $r > 0$, $x = r$ is a local minimum of V and therefore a stable fixed point, while $x = 0$ is a local maximum of V, and therefore an unstable fixed point of the equation.

c) The potential is

$$V(x) = -r\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6}.$$

Let us first try to understand which are the qualitatively different ranges of the parameter r.

To find the stationary points of V (that is, the fixed points of the equation), you must solve the equation

$$rx + x^3 - x^5 = 0. \tag{1}$$

One solution is always $x = 0$. For $x \neq 0$, Eq. (1) becomes

$$r + x^2 - x^4 = 0. \tag{2}$$

Setting $y = x^2$, this equation is equivalent to

$$y^2 - y - r = 0, \tag{3}$$

which has the solutions

$$y = \frac{1 \pm \sqrt{1+4r}}{2}. \tag{4}$$

Only real and positive solutions y of Eq. (3) give rise to real solutions of Eq. (2). For $r < -1/4$, the two solutions (4) of Eq. (3) are not real. For $-1/4 < r < 0$, both solutions (4) of Eq. (3) are real and positive. We therefore get four solutions x of Eq. (2) in this parameter range:

$$x = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}}. \tag{5}$$

For $r > 0$,

$$\frac{1 + \sqrt{1+4r}}{2} > 0,$$

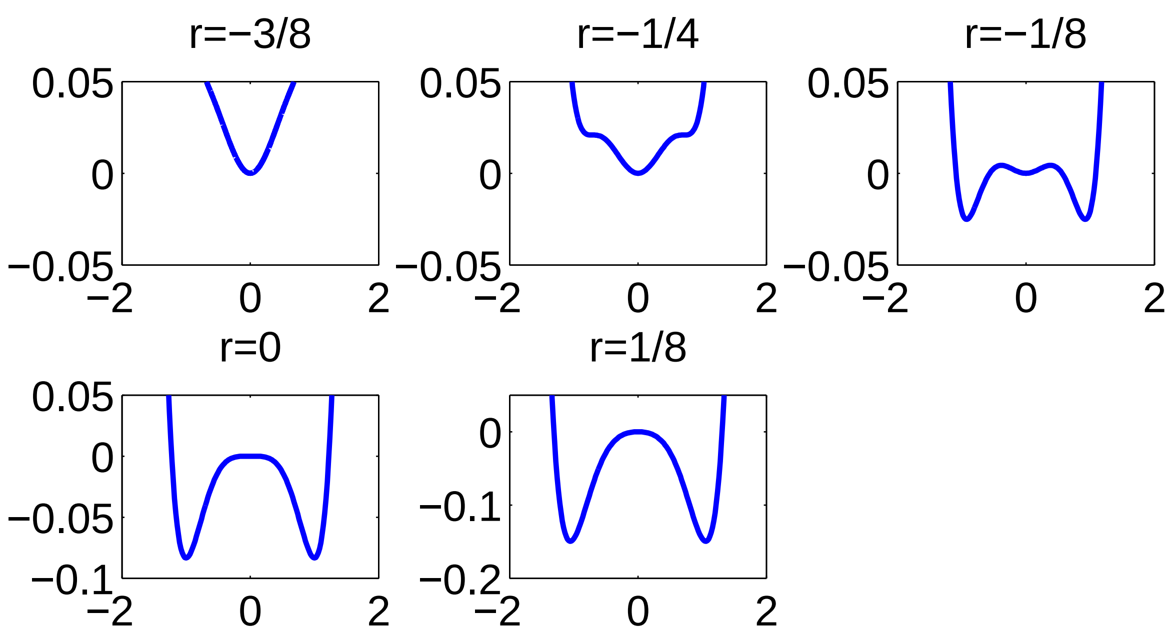
but

$$\frac{1+\sqrt{1-4r}}{2} < 0,$$

so we obtain two solutions x of Eq. (2) for $r > 0$:

$$x = \pm\sqrt{\frac{1+\sqrt{1+4r}}{2}}. \tag{6}$$

The three qualitatively different parameter ranges, therefore, are $r < -1/4$, $-1/4 < r < 0$, and $r > 0$. Here are the plots of V for values of r in each of these three ranges, as well as $r = -1/4$ and $r = 0$ (the values at which there are bifurcations):



There are two saddle-node bifuractions at $r = -1/4$, and a subcritical pitchfork bifurcation at $r = 0$.

2 Newton's force law is

$$m\ddot{x} = -mg - k\left(\sqrt{x^2+h^2} - L\right)\frac{x}{\sqrt{x^2+h^2}} - b\dot{x}, \tag{7}$$

as derived in class. We define $T = b/k$ now, and

$$y(\tau) = x(\tau T).$$

Primes denote derivatives with respect to τ:

$$y'(\tau) = T\dot{x}(\tau T), \quad y''(\tau) = T^2\ddot{x}(\tau T).$$

Using these equations in Eq. (7), we find:

$$\frac{m}{T^2}y = -mg - k\left(\sqrt{y^2+h^2} - L\right)\frac{y}{\sqrt{y^2+h^2}} - \frac{b}{T}y'.$$

We insert $T = b/k$:

$$\frac{mk^2}{b^2}y = -mg - k\left(\sqrt{y^2+h^2} - L\right)\frac{y}{\sqrt{y^2+h^2}} - ky'$$

Now let $b \to \infty$:

$$0 = -mg - k\left(\sqrt{y^2+h^2} - L\right)\frac{y}{\sqrt{y^2+h^2}} - ky'$$

Solve for y':

$$y' = -\frac{mg}{k} - \left(\sqrt{y^2+h^2} - L\right)\frac{y}{\sqrt{y^2+h^2}}$$

Now set $z = y/h$:

$$hz' = -\frac{mg}{k} - \left(\sqrt{h^2z^2+h^2} - L\right)\frac{zh}{\sqrt{h^2z^2+h^2}}$$

Divide both sides by h:

$$z' = -\frac{mg}{kh} - \left(\sqrt{z^2+1} - \frac{L}{h}\right)\frac{z}{\sqrt{z^2+1}}$$

Simplify:

$$z' = -s - \left(1 - \frac{r}{\sqrt{z^2+1}}\right)z$$

with

$$s = \frac{mg}{kh}, \quad r = \frac{L}{h}.$$

This is precisely the equation that we got in class, except that the right-hand side has been multiplied by s.

3 This is a silly question in a way, but I asked it so that you think about the catastrophe surfaces one more time:

$$\dot{x} = x^2 + r$$

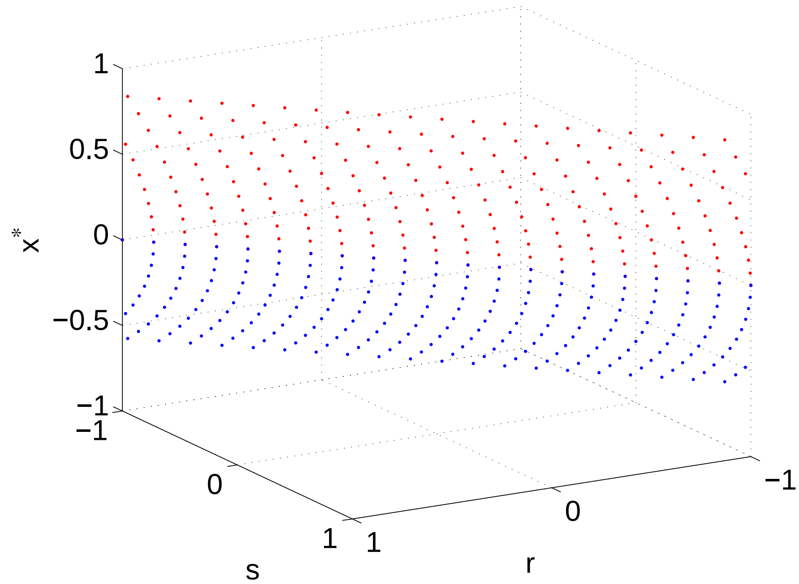
is an equation in which there is a saddle-node bifurcation when r passes through $r_c = 0$: For $r < 0$ there are two fixed points, for $r > 0$ there is none. Add a second parameter s:

$$\dot{x} = x^2 + r + s.$$

(The reason why this is silly is that it is unnatural to think of "$r+s$" as the sum of two independent parameters. There is only one parameter here, namely $r+s$.) There are two fixed points if $r+s < 0$, none if $r+s > 0$. The catastrophe surface is a surface in (r,s,x^*)-space. Above (r,s), you plot the fixed point(s), if any. So you plot

$$\pm\sqrt{-r-s}$$

above (r,s) if $r+s \leq 0$, and nothing if $r+s > 0$. Furthermore, the negative fixed point is stable, and the positive is unstable. The surface looks like a bent piece of paper:



(Unstable fixed points are indicated in red, and stable ones in blue.)

4 The function $f(\theta) = \sin(a\theta)$ must be periodic with period 2π. Therefore a must be an integer.

5 4.1.2. The fixed points are solutions of

$$1 + 2\cos\theta = 0,$$

that is:

$$\theta = \arccos(-1/2) + 2\pi k, \quad k \text{ integer},$$

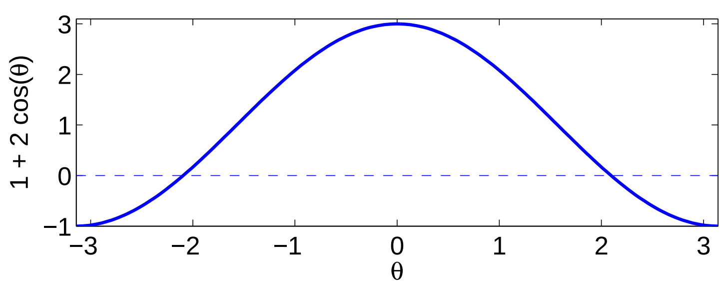
and

$$\theta = -\arccos(-1/2) + 2\pi k, \quad k \text{ integer}.$$

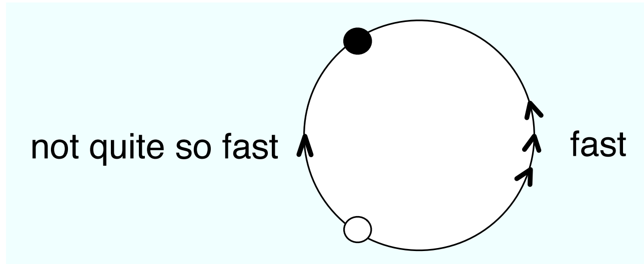
Since $\arccos(-1/2) = 2\pi/3$, the solutions are:

$$\theta = \pm\frac{2}{3}\pi + 2\pi k, \quad k \text{ integer}.$$

The function $1 + \cos(2\theta)$ looks like this on the interval $[-\pi, \pi]$:



Thus the fixed point $2\pi/3$ is stable, and $-2\pi/3$ is unstable. The phase portrait looks like this:



The motion is fastest near $\theta = 0$, where $\dot{\theta} = 3$. It is not quite so fast near $\theta = -\pi$, there $\dot{\theta} = -1$.

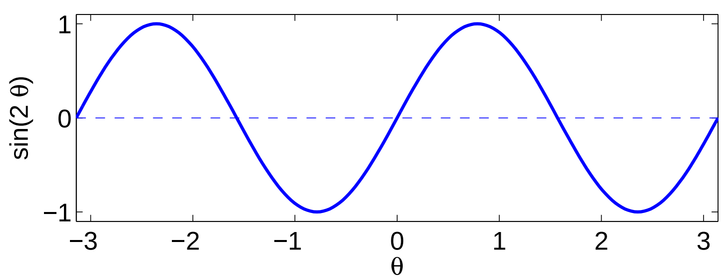
4.1.3. The fixed points are solutions of

$$\sin(2\theta) = 0,$$

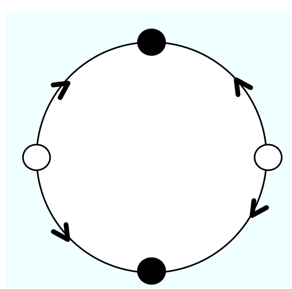
that is:

$$\theta = k\frac{\pi}{2}, \quad k \text{ integer.}$$

The function $\sin(2\theta)$ looks like this on the interval $[-\pi, \pi]$:



So there are two stable fixed points, at $-\pi/2$ and $\pi/2$ (plus an integer multiple of 2π), and two unstable ones, at 0 and π (plus an integer multiple of 2π). The phase portrait looks like this:



4.1.5. Things simplify a lot if you first observe that

$$\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

Therefore the fixed points are solutions of

$$\sin(\theta + \pi/4) = 0,$$

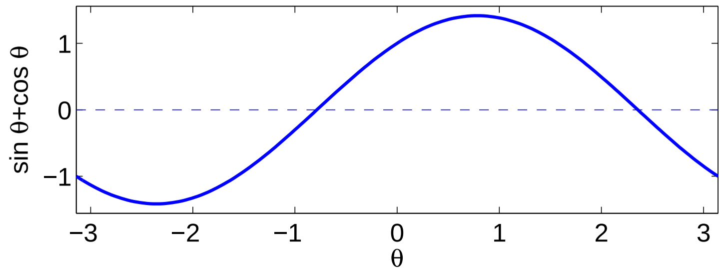
that is:

$$\theta = -\frac{\pi}{4} + k2\pi, \quad k \text{ integer},$$

and

$$\theta = \frac{3}{4}\pi + k2\pi, \quad k \text{ integer}.$$

The function $f(\theta) = \sin\theta + \cos\theta$ looks like this:



So the fixed point $\theta = 3\pi/4 + k2\pi$ is stable, and the fixed point $\theta = -\pi/4 + k2\pi$ is unstable.