

Solutions for Homework 4

1 Solution 1: The hour hand is a uniform oscillator with period $T = 1 = 60$ minutes. The minute hand is a uniform oscillator with period $T = 1$ minute. The two oscillators are described by

$$\dot{\theta}_1 = \frac{2\pi}{60}$$

and

$$\dot{\theta}_2 = 2\pi$$

if time is measured in minutes. The phase difference satisfies

$$\frac{d}{dt}(\theta_2 - \theta_1) = 2\pi \left(1 - \frac{1}{60}\right) = 2\pi \frac{59}{60}$$

Therefore the phase difference is itself a uniform oscillator, with period

$$\frac{60}{59}.$$

Thus the two hands are aligned again after $60/59$ minutes, or $3600/59 \approx 61.0169$ seconds.

Solution 2: After one minute, the minute hand returns to where it started, but the hour hand has advanced by $1/60$ of a rotation. The minute hand needs $1/60$ of a minute, that is, one second, to get there. But in that time, the hour hand has advanced $1/60$ of $1/60$ of a rotation. The minute hand needs $1/60$ of $1/60$ of a minute to get there. And so on. So the time it takes for the minute hand to reach the hour hand is

$$\sum_{k=0}^{\infty} \frac{1}{60^k},$$

which is a geometric series. Its sum is

$$\frac{1}{1 - 1/60} = \frac{60}{59}$$

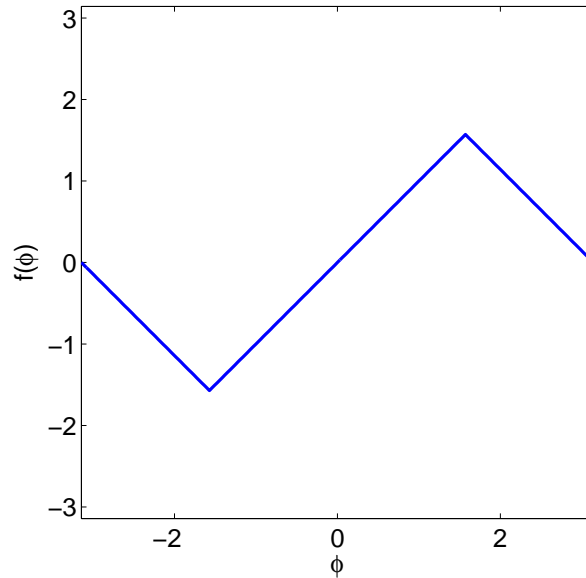
minutes.

We arrive at the curious conclusion that Solution 1 constitutes a slick proof of the formula

$$\sum_{k=0}^{\infty} \frac{1}{60^k} = \frac{60}{59}.$$

(This formula was proved quite differently in your Calculus II course.)

2 a)



b) The difference ϕ in phases between the dictator and the firefly satisfies

$$\dot{\phi} = \Omega - \omega - Af(\phi).$$

To compute the fixed points, we have to solve the equation

$$f(\phi) = \frac{\Omega - \omega}{A}.$$

To see what are the solutions of this equation, draw the horizontal line at

$$f = \frac{\Omega - \omega}{A}$$

into the above graph, and ask where it intersects the graph.

If

$$\left| \frac{\Omega - \omega}{A} \right| > \frac{\pi}{2}, \quad (1)$$

there are no fixed points. If

$$\left| \frac{\Omega - \omega}{A} \right| < \frac{\pi}{2}, \quad (2)$$

there are two fixed points. One is

$$\phi_1^* = \frac{\Omega - \omega}{A},$$

and the other at

$$\phi_2^* = \pi - \frac{\Omega - \omega}{A} \quad (\text{modulo } 2\pi).$$

To see which fixed point is stable, and which is unstable, we have to check the sign of

$$\frac{d}{d\phi} (\Omega - \omega - Af(\phi)) = -Af'(\phi)$$

at the fixed points. If $A > 0$, then ϕ_1^* is stable, and if $A < 0$, then ϕ_2^* is stable.

In particular, think about the case when

$$\left| \frac{\Omega - \omega}{A} \right| \approx 0.$$

Then the two fixed points lie near 0 and π (modulo 2π), which corresponds to approximate synchrony and anti-synchrony. (“Anti-synchrony” means that the firefly and the dictator are off by half a period.) If $A > 0$ (the firefly tries to catch up with the dictator), then there will be approximate synchrony in the long run. This is true regardless of whether $\Omega > \omega$ or $\Omega < \omega$. If, on the other hand, $A < 0$ (the firefly tries to fall behind further when it is close to caught up with the dictator), then there will be approximate anti-synchrony in the long run.