Homework 5, Solutions

3. p. 143, problem 5.2.13.

a) As a two-dimensional linear system, the equations look like this:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\frac{b}{m}y - \frac{k}{m}x.\end{aligned}$$

The matrix is

$$A = \frac{1}{m}\begin{bmatrix} 0 & m \\ -k & -b \end{bmatrix}$$

b) The trace of A is

$$\tau = -\frac{b}{m} < 0,$$

and the determinant is

$$\Delta = \frac{k}{m} > 0.$$

This implies that the origin is either a stable node, or a stable spiral. It is a stable spiral if

$$\Delta > \frac{\tau^2}{4},$$

which means

$$\frac{k}{m} > \frac{b^2}{4m^2},$$

or

$$b < 2\sqrt{km}.$$

It is a stable node if

$$b > 2\sqrt{km}.$$

The borderline case

$$b = 2\sqrt{km},$$

will be discussed later.

Let us assume

$$b > 2\sqrt{km},$$

so that the origin is a stable node, and compute the eigenvalues and eigenvectors of A. The eigenvalues are solutions of

$$(\lambda - 0)(\lambda + b) + km = 0,$$

so

$$\lambda^2 + b\lambda + km = 0,$$

or

$$\lambda = \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4km}}{2}$$

Both of these are negative real numbers; the smaller of the two if λ_-.

The eigenvector associated with an eigenvalue λ is a solution

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

of

$$\begin{aligned} y &= \lambda x, \\ -(k/m)x - (b/m)y &= \lambda y. \end{aligned}$$

These two equations must be equivalent to each other if λ is an eigenvalue (otherwise $A - \lambda I$ would not be singular). Therefore it suffices to look at the first of the two equations,

$$y = \lambda x.$$

So the eigenvectors associated with the eigenvalue λ are the nonzero multiples of the vector

$$\begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

So the fast direction of the node at the origin is

$$\begin{bmatrix} 1 \\ \lambda_- \end{bmatrix},$$

and the slow direction is

$$\begin{bmatrix} 1 \\ \lambda_+ \end{bmatrix}.$$

All trajectories approach the origin tangentially to the slow direction, *except* for those that come in *precisely* in the direction of the fast direction.

When $b = 2\sqrt{km}$, there is only one eigenvalue, namely

$$\lambda = -b/2,$$

and the above discussion implies that there is only one eigendirection, namely the direction of the vector

$$\begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

All trajectories approach the origin tangentially to this direction. To understand why this is the case, you have to recall (from Math 38) that the general solution of

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

is, in this case,

$$c_1 \begin{bmatrix} 1 \\ \lambda \end{bmatrix} t e^{\lambda t} + c_2 \underline{w} e^{\lambda t}, \tag{1}$$

where $\underline{w}$ is a generalized eigenvector, that is, a solution of

$$(A - \lambda I)\underline{w} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

The crucial point to observe now is that even though (1) is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad \text{and} \quad \underline{w},$$

the component in the direction

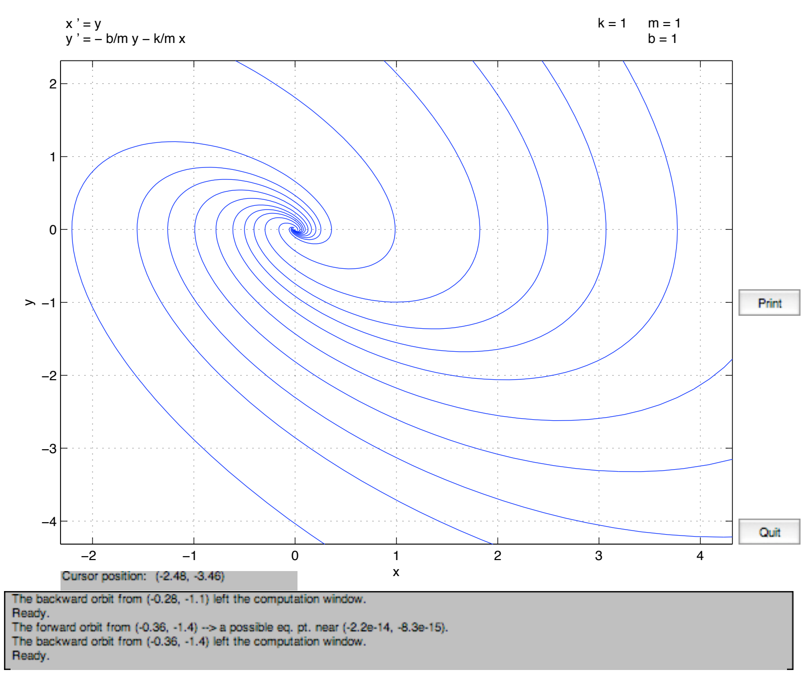
$$\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

dominates more and more heavily as $t \to \infty$.

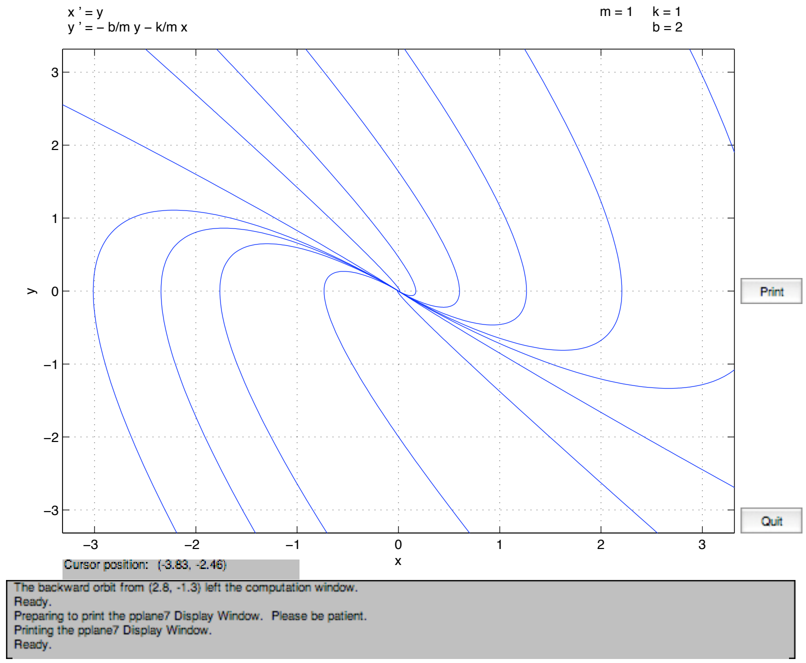
So if $b = 2\sqrt{km}$, the origin is a degenerate node with only one possible incoming direction.

Based on this information, you can draw, qualitatively, what the phase plane picture looks like, but this is difficult for me to do electronically unless I resort to software that draws phase portrait. This is what I did here (but you should know how to qualitatively sketch the pictures without such software). I used the Matlab code `pplane7`. (If you are interested, you'll find it instantly if you google "pplane7".) I took $k = m = 1$, and $b = 1$ (stable spiral), $b = 2$ (degenerate node), and $b = 3$ (stable node).

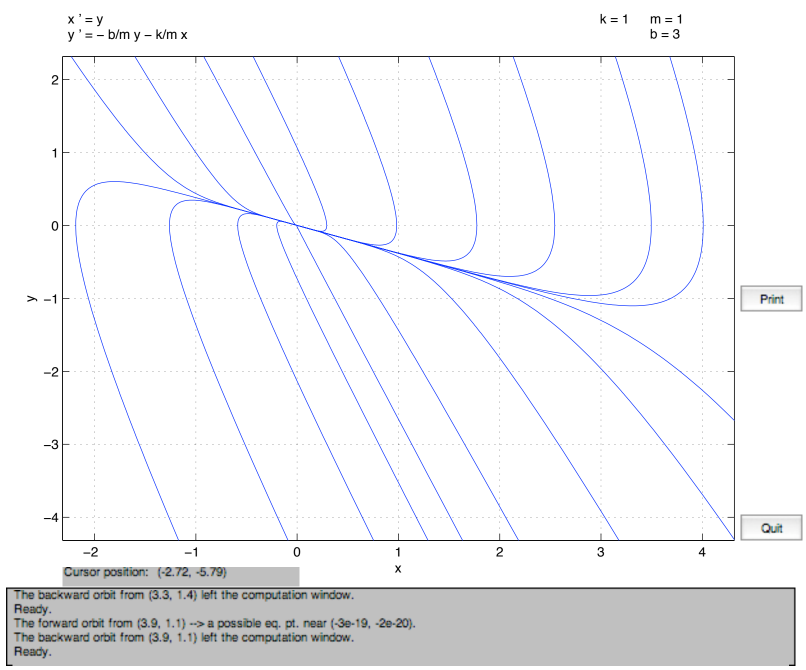
$b = 1$:



$b = 2$:



$b = 3$:



(c) $b = 1$ is "underdamped" (that is, the damping does not eliminate oscillations), $b = 3$ is "overdamped" (the damping is more than enough to eliminate oscillations), and $b = 2$ is "critically damped" (the damping is just barely enough to eliminate oscillations).