Homework 6

Due: Friday October 23

Think about the Lotka-Volterra model of competition again. In class, I worked out one particular example, with very specific coefficients. Now think, in general, about

$$\dot{X} = rX\left(1 - \frac{X}{K} - aY\right), \quad (1)$$

$$\dot{Y} = sY\left(1 - \frac{Y}{L} - bX\right), \quad (2)$$

where X is the number of rabbits, Y the number of sheep, K is the carrying capacity for the rabbits in the absence of sheep, L is the carrying capacity for the sheep in the absence of rabbits, and $a > 0$, $b > 0$.

a) Define x to be X, measured in units of K, and y to be Y, measured in units of Y, that is:

$$x = \frac{X}{K} \quad \text{and} \quad y = \frac{Y}{L}.$$

Show that with this notation, our equations take the form

$$\dot{x} = rx(1 - x - \alpha y), \quad (3)$$

$$\dot{y} = sy(1 - y - \beta x), \quad (4)$$

with $\alpha > 0$ and $\beta > 0$. What are α and β, expressed in terms of parameters in the original equations (1) and (2)? So the conclusion is that it is no loss of generality to assume that the carrying capacities for both species equal 1.

b) As in class, there are three fixed points corresponding to one or both species' going extinct:

$$(0, 0), \quad (0, 1), \quad (1, 0).$$

Show that $(0, 0)$ is always an unstable node.

c) An interesting question is whether there is a fourth fixed point (x^*, y^*) in which the two species coexist. Show that there are exactly two cases in which such a fixed point exists, and $x^* > 0$ and $y^* > 0$:[1]

Weak inter-species competition: $\alpha < 1$ and $\beta < 1$

Strong inter-species competition: $\alpha > 1$ and $\beta > 1$

I will call (x^*, y^*) the "coexistence fixed point".

d) Show that in the coexistence fixed point, the Jacobi matrix is

$$J = \begin{bmatrix} -rx^* & -\alpha r x^* \\ -\beta s y^* & -sy^* \end{bmatrix}.$$

[1] Obviously, if $x^* < 0$ and/or $y^* < 0$, the fixed point is of no biological interest.

(Hint: This becomes a very brief computation if you use the system of linear equations that x^* and y^* satisfy, refraining from solving that system.)

e) For the case of strong inter-species competition, conclude from d) that the coexistence fixed point is a saddle point; thus coexistence is unstable in that case. (The competitive exclusion principle applies.)

f) For the case of strong inter-species competition, show that $(1, 0)$ and $(0, 1)$ are stable fixed points.

g) For the case of weak inter-species competition, conclude from d) that the coexistence fixed point is stable.

h) For the case of weak inter-species competition, show that $(1, 0)$ and $(0, 1)$ are saddle points.

i) Make a qualitative sketch of the phase portrait for the case of weak inter-species competition.