# Homework 6, Solutions

a) $\alpha = aL$ and $\beta = bK$. Eq. (3) is obtained from (1) by dividing both sides by $K$. Eq. (4) is obtained from (2) by dividing both sides by $L$.

b) $(x^*, y^*)$ is a fixed point of (3), (4) if and only if

$$x^* = 0 \text{ and } y^* = 0, \tag{1}$$

or

$$1 - x^* - \alpha y^* = 0 \text{ and } y^* = 0, \tag{2}$$

or

$$x^* = 0 \text{ and } 1 - y^* - \beta x^* = 0, \tag{3}$$

or

$$1 - x^* - \alpha y^* = 0 \text{ and } 1 - y^* - \beta x^* = 0. \tag{4}$$

(1), (2), and (3) correspond to the three fixed points $(0,0)$, $(1,0)$, and $(0,1)$. The Jacobi matrix in $(0,0)$ is

$$J = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}.$$

This matrix has two positive eigenvalues, therefor the origin is an unstable node.

c) To find a fixed point $(x^*, y^*)$ in which the two species coexist, we have to solve the linear system (4). In matrix-vector form:

$$\begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5}$$

The system is non-singular if $\alpha\beta \neq 1$. In that case, Cramer's Rule tells us that

$$x^* = \frac{1-\alpha}{1-\alpha\beta} \quad \text{and} \quad y^* = \frac{1-\beta}{1-\alpha\beta}.$$

Evidently the only case in which both $x^*$ and $y^*$ are positive are $\alpha < 1$ and $\beta < 1$, or $\alpha > 1$ and $\beta > 1$.

Let's think about the case when (5) is a singular system, that is, $\alpha\beta = 1$. In that case, the first of the two equations in (5) can be multiplied by $\beta$ (which is nonzero since $\alpha\beta = 1$) to yield

$$\beta x^* + \alpha\beta y^* = \beta.$$

Since $\alpha\beta = 1$, this is equivalent to

$$\beta x^* + y^* = \beta.$$

So Eqs. (5) can be re-written as

$$\begin{aligned} \beta x^* + y^* &= \beta, \\ \beta x^* + y^* &= 1. \end{aligned}$$

So if $\beta \neq 1$, then the system is not solvable — there is no co-existence fixed point. If $\beta = 1$, then also $\alpha = 1$ (since we are assuming $\alpha\beta = 1$ here), and Eqs. (5) amount to

$$x^* + y^* = 1.$$

So in the very special case $\alpha = \beta = 1$, there is in fact a whole line of coexistence fixed points. I will ignore this case here. (I had not thought of it when I wrote the problem!)

d) The Jacobi matrix is

$$J = \begin{bmatrix} r - 2rx^* - \alpha ry^* & -\alpha rx^* \\ -\beta sy^* & s - 2sy^* - \beta sx^* \end{bmatrix}. \tag{6}$$

Using Eqs. (4), this becomes

$$J = \begin{bmatrix} -rx^* & -\alpha rx^* \\ -\beta sy^* & -sy^* \end{bmatrix}. \tag{7}$$

e) – h) The trace is

$$\tau = -rx^* - sy^*,$$

and the determinant is

$$\Delta = rs(1 - \alpha\beta)x^*y^*.$$

Let us consider the coexistence fixed point first ($x^* > 0$, $y^* > 0$). If $\alpha > 1$, $\beta > 1$ (strong competition), then $\Delta < 0$, so the coexistence fixed point is a saddle, and the competitive exclusion principle applies. If $\alpha < 1$, $\beta < 1$ (weak competition), then $\Delta > 0$, $\tau < 0$, and therefore the coexistence fixed point is either a stable node, or a stable spiral. Could it be a stable spiral? That is, could $\Delta$ be greater than $\tau^2/4$? This would mean

$$rs(1 - \alpha\beta)x^*y^* > \frac{(rx^* + sy^*)^2}{4}.$$

A little bit of algebra shows that this inequality is equivalent to

$$-4rs\alpha\beta x^*y^* > (rx^* - sy^*)^2,$$

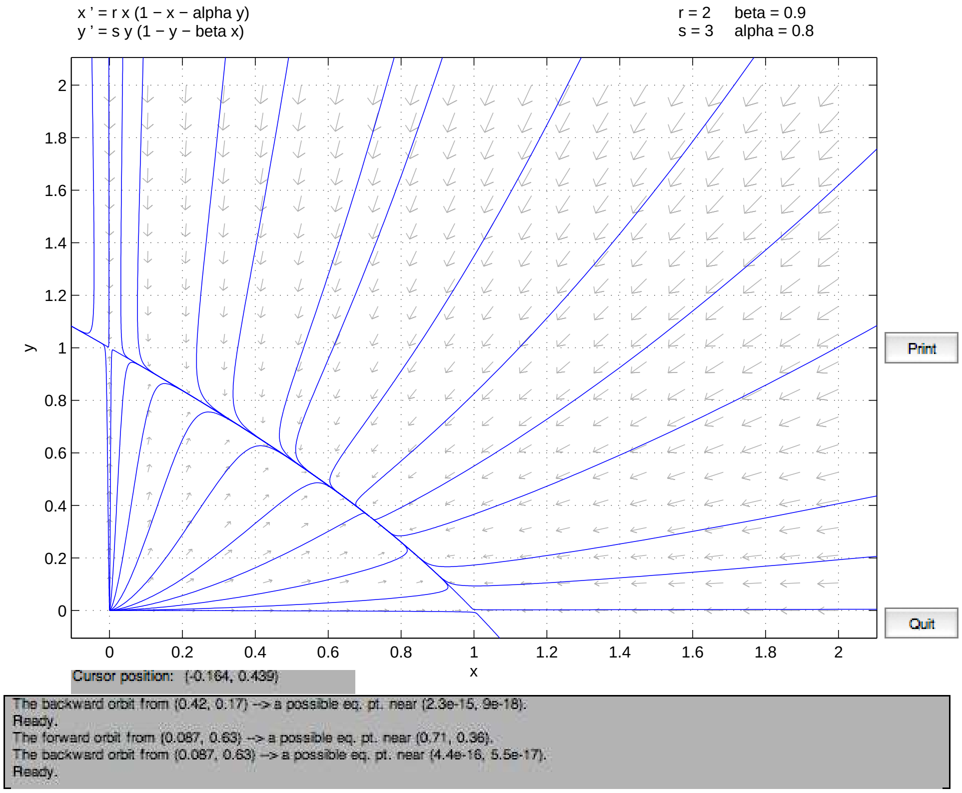
which is evidently impossible. So the coexistence fixed point is a stable node in the case of weak competition.

Now let's think about the extinction fixed point $(1,0)$. In this point, the Jacobi matrix is

$$J = \begin{bmatrix} -r & -\alpha r \\ 0 & (1 - \beta)s \end{bmatrix}.$$

(Remember that you have to use (6) here, not (7).) So if $\beta < 1$, then $J$ has a positive and a negative eigenvalue, and therefore $(1,0)$ is a saddle. If $\beta > 1$, then $J$ has two negative eigenvalues, and therefore $(1,0)$ is a stable node. The extinction fixed point $(0,1)$ is analyzed analogously.

i) Again I will assert that I could do this without a computer, just not in electronic form... This is my excuse for using Matlab, and here is the phase portrait, drawn by `pplane7`, for $r = 2$ $s = 3$, $\alpha = 0.8$, $\beta = 0.9$:



x ' = r x (1 − x − alpha y)
y ' = s y (1 − y − beta x)

r = 2 beta = 0.9
s = 3 alpha = 0.8

Cursor position: (-0.164, 0.439)

```
The backward orbit from (0.42, 0.17) --> a possible eq. pt. near (2.3e-15, 9e-18).
Ready.
The forward orbit from (0.087, 0.63) --> a possible eq. pt. near (0.71, 0.36).
The backward orbit from (0.087, 0.63) --> a possible eq. pt. near (4.4e-16, 5.5e-17).
Ready.
```