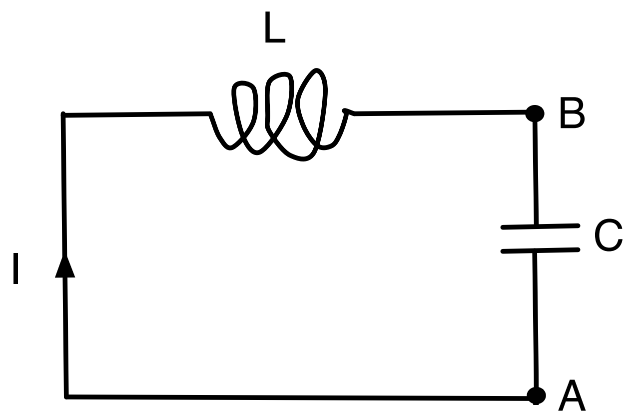
Homework 7

Due: Friday, October 30

1. p. 228, problem 7.1.2.

2. p. 228, problem 7.1.5.

3. (preparation for problem 4) Consider an electrical circuit like this:



This circuit contains a capacitor (labeled C) and a coil (labeled L). We assume that there is no resistance in the circuit. Let $I = I(t)$ be the current in the clockwise direction through the circuit. (So $I < 0$ means current in the counterclockwise direction.) The current is the same at any location in the circuit. The electrons drift in the direction opposite to the direction of the current. (This is because they are negatively charged.)

The electrical potential drops across the coil and the capacitor, but is constant everywhere else. Let V_A and V_B be the electrical potentials at the points A and B. The work needed to move a positive unit test charge from A to B is $V_B - V_A$. Notice that $V_B - V_A > 0$ means that the potential difference opposes current from A to B, while $V_B - V_A < 0$ means that it facilitates such current.

If a positive current flows through the circuit, the electrons drift in the counterclockwise direction and accumulate near A, namely on the lower plate of the capacitor. This makes it harder to move a positive test charge from A to B. Positive I therefore causes a rise in $V_B - V_A$. The relation between this rise in $V_B - V_A$ and I is linear:

$$C\frac{d}{dt}(V_B - V_A) = I\ , \tag{1}$$

where $C > 0$ is the capacitance.

Even though resistors play no role in this problem, let me next remind you of how they work. If the coil in the picture above were a resistor, a potential difference would be needed to drive a current across it. The relation between the needed potential difference and the current is again linear:

$$V_B - V_A = -RI\ ,$$

with $R > 0$. (Notice the minus sign. To drive a positive current across the resistor, we need a facilitating potential difference, that is, $V_A > V_B$.)

The coil in our problem is assumed to have no resistance, but inductance. This means that a change of current sets up a magnetic field opposing the change. A voltage difference is needed to maintain the change:

$$V_B - V_A = -L\frac{dI}{dt} \ , \tag{2}$$

with $L > 0$. The constant L is called the inductance of the coil. (Again, there is a minus sign. To drive an increasing current across the inductor, we must have a facilitating voltage difference, that is, $V_A > V_B$.)

Define $V = V_A - V_B$. Then equations (1) and (2) above are

$$\dot{V} = -I/C, \tag{3}$$

$$\dot{I} = V/L. \tag{4}$$

Draw the phase portrait for Eqs. (1) and (2). Physically, what does the phase portrait tell you? Write down a single second order equation just for I.

4. Do problem 7.1.6 on p. 228. Show that you get the van der Pol equation with $F(x) = x^3/3 - x$.

5. p. 231, problem 7.3.3.