Solutions for Homework 7

3. The equations can be written like this:

$$\begin{bmatrix} \dot{V} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}.$$

Armed with Math 38 knowledge (or equivalent), you can write down the solution explicitly:

$$\begin{aligned} V(t) &= V(0)\cos\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{L}{C}} I(0)\sin\left(\frac{t}{\sqrt{LC}}\right), \\ I(t) &= I(0)\cos\left(\frac{t}{\sqrt{LC}}\right) + \sqrt{\frac{C}{L}} V(0)\sin\left(\frac{t}{\sqrt{LC}}\right). \end{aligned}$$

The trajectories are ellipses. To see this, consider for instance the case $I(0) = 0$. Then

$$\begin{aligned} V(t) &= V(0)\cos\left(\frac{t}{\sqrt{LC}}\right), \\ I(t) &= \sqrt{\frac{C}{L}} V(0)\sin\left(\frac{t}{\sqrt{LC}}\right). \end{aligned}$$

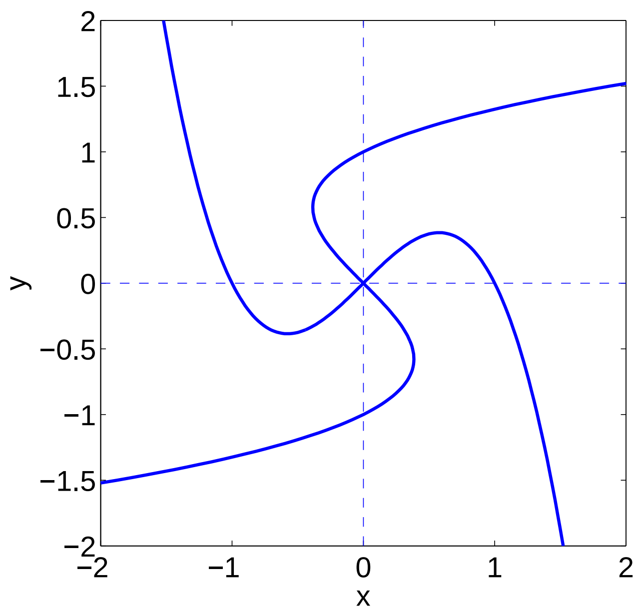
This describes an ellipse, with one half-axis equal to $V(0)$, the other equal to $\sqrt{C/L} \times V(0)$. The second order equation for I is

$$\ddot{I} = -\frac{I}{CL}.$$

5. 7.3.3 The only fixed point is $(0,0)$. There are many ways of seeing this. One is to plot the nullclines, given by

$$y = x - x^3 \quad \text{and} \quad x = y^3 - y.$$

Here is a plot of these nullclines:



It is easy to convince yourself that Matlab got it right, and there are no intersections other than the origin; the key is to yourself that the maximum of $y = x - x^3$ between $x = 0$ and $x = 1$ is

$$\frac{2}{3}\frac{1}{\sqrt{3}},$$

which is < 1. This implies that there cannot be any fixed point in the right upper quadrant, other than the origin. By symmetry, the other three quadrants can't contain any fixed points other than the origin.

Let us examine what kind of fixed point the origin is. The Jacobi matrix at the origin is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

So the determinant is $\Delta = 2$, and the trace is $\tau = 2$. Since $\Delta > 0$, $\tau > 0$, and $2 = \Delta > \tau^2/4 = 1$, the origin is an unstable spiral.

Consider now the square with side length $2a$, centered at the origin, where $a > 0$. Its right edge is defined by $x = a$, $-a \leq y \leq a$. Does the vector field point to the left along this edge? So we have to ask whether

$$a - y - a^3 < 0$$

for $-a \leq y \leq a$. When $-a \leq y \leq a$, the largest that $a - y - a^3$ could be is $2a - a^3 = a(2 - a^2)$, so if $a > \sqrt{2}$, then indeed the vector field points to the left along the right edge. By symmetry, it must then also point to the right along the left edge, downwards along the upper edge, and upwards along the lower edge. (If you are not convinced of the symmetry argument, just give arguments analogous to the one above for the other three edges.)

Define R to be a square, centered at the origin, with side length greater than $2\sqrt{2}$, with a small disk around the origin cut out. If the disk is small enough, the vector field will point into R from its boundary, since $(0,0)$ is an unstable spiral. So R is a trapping region.