Homework 8, Solutions

page 285, problem 8.1.10:

a) The average size S of the trees grows logistically, but with a carrying capacity that depends on the health of the forest: The carrying capacity is

$$\frac{E}{K_E}K_S,$$

where K_S denotes the equilibrium value of S when E is at its "carrying capacity" K_E. Typically, E will be less than K_E, and as a result the carrying capacity of S will be less than K_S. At the same time, E also follows a logistic law, with "carrying capacity" K_E, but in addition it is driven down by the term

$$-P\frac{B}{S}.$$

Note that the ratio B/S is proportional to the density with which budworms populate trees.

b) "Non-dimensionalize" is a slightly problematic word here. Which units should be assign to E, for instance? But non-dimensionalization really means writing the equation in such a way that the number of parameters that appear is as small as possible. This we can do. We simply write:

$$t = t_0\tilde{t}, \quad S = S_0\tilde{S}, \quad E = E_0\tilde{E},$$

where t_0, S_0, and E_0 should be thought of as the units in which we will measure t, S, and E. Then

$$\frac{d\tilde{S}}{d\tilde{t}} = \frac{t_0}{S_0}\frac{dS}{dt} = \frac{t_0}{S_0}\left[r_s S\left(1-\frac{S}{K_S}\frac{K_E}{E}\right)\right] =$$

$$\frac{t_0}{S_0}\left[r_s S_0\tilde{S}\left(1-\frac{S_0\tilde{S}}{K_S}\frac{K_E}{E_0\tilde{E}}\right)\right] = t_0\left[r_s\tilde{S}\left(1-\frac{S_0\tilde{S}}{K_S}\frac{K_E}{E_0\tilde{E}}\right)\right] \tag{1}$$

and

$$\frac{d\tilde{E}}{d\tilde{t}} = \frac{t_0}{E_0}\frac{dE}{dt} = \frac{t_0}{E_0}\left[r_E E\left(1-\frac{E}{K_E}\right) - P\frac{B}{S}\right] =$$

$$\frac{t_0}{E_0}\left[r_E E_0\tilde{E}\left(1-\frac{E_0\tilde{E}}{K_E}\right) - P\frac{B}{S_0\tilde{S}}\right] = t_0\left[r_E\tilde{E}\left(1-\frac{E_0\tilde{E}}{K_E}\right) - P\frac{B}{S_0E_0\tilde{S}}\right] \tag{2}$$

To make the equations as simple as possible, we set

$$S_0 = K_S, \quad E_0 = K_E, \quad t_0 = \frac{1}{r_s}.$$

Then Eqs. (1) and (2) become:

$$\begin{aligned}
\frac{d\tilde{S}}{d\tilde{t}} &= \tilde{S}\left(1-\frac{\tilde{S}}{\tilde{E}}\right), \\
\frac{d\tilde{E}}{d\tilde{t}} &= \frac{r_E}{r_S}\tilde{E}\left(1-\tilde{E}\right) - \frac{PB}{K_S K_E r_s}\frac{1}{\tilde{S}}.
\end{aligned}$$

Now we set

$$\gamma = \frac{r_S}{r_E}, \quad \beta = \frac{PB}{K_S K_E r_S},$$

and drop the tildes:

$$\begin{aligned}\frac{dS}{dt} &= S\left(1 - \frac{S}{E}\right), \\ \frac{dE}{dt} &= \frac{1}{\gamma}E(1-E) - \frac{\beta}{S}.\end{aligned}$$

c) Let's work with the non-dimensionalized system. The $dS/dt = 0$ nullcline is given by

$$S = 0 \quad \text{or} \quad S = E.$$

The $dE/dt = 0$ nullcline is given by

$$S = \frac{\beta\gamma}{E(1-E)}.$$

Fixed points are points where the graphs of

$$f(E) = E$$

and

$$g(E) = \frac{\beta\gamma}{E(1-E)}$$

intersect. The graph of g has vertical asymptotes at $E = 0$ and $E = 1$, and its minimum between 0 and 1 occurs at $E = 1/2$, where the value is $g(1/2) = 4\beta\gamma$. The graph of g intersects with the graph of f if and only if $g(1/2) < f(1/2)$, that is,

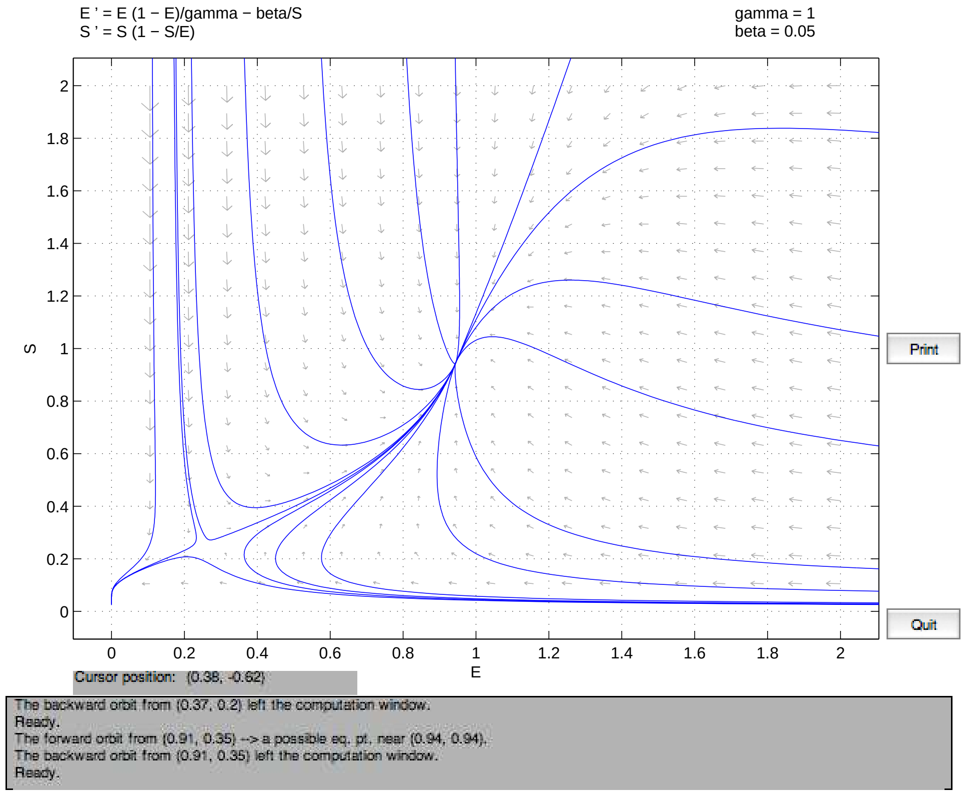
$$4\beta\gamma < 1/2$$

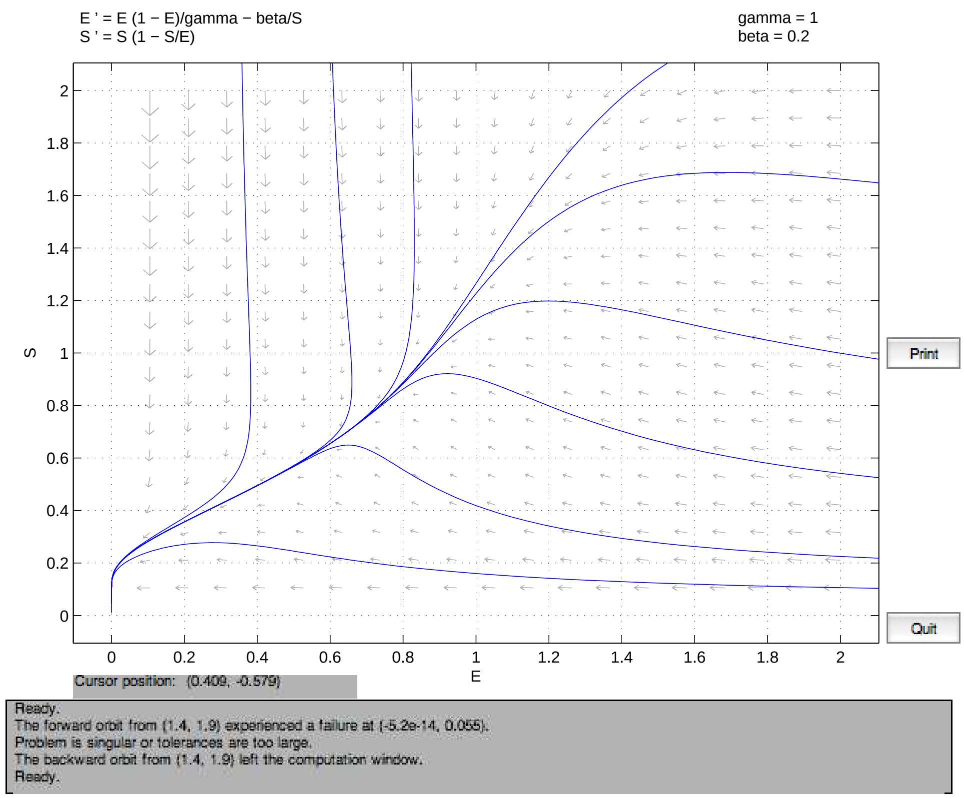
or

$$8\beta\gamma < 1.$$

If we think of β (which is proportional to B) as the bifurcation parameter, there are two fixed points for $\beta < 1/(8\gamma)$, and none for $\beta > 1/(8\gamma)$. A saddle-node bifurcation occurs at $\beta = \beta_c = 1/(8\gamma)$.

d) Here are phase portraits for $\gamma = 1$, $\beta = 0.05$ (below the bifurcation) and $\beta = 0.2$ (above the bifurcation):


E ' = E (1 − E)/gamma − beta/S
S ' = S (1 − S/E)
gamma = 1
beta = 0.05
S
E
Print
Quit
Cursor position: (0.38, -0.62)
The backward orbit from (0.37, 0.2) left the computation window.
Ready.
The forward orbit from (0.91, 0.35) --> a possible eq. pt. near (0.94, 0.94).
The backward orbit from (0.91, 0.35) left the computation window.
Ready.


E ' = E (1 − E)/gamma − beta/S
S ' = S (1 − S/E)
gamma = 1
beta = 0.2
S
E
Print
Quit
Cursor position: (0.409, -0.579)
Ready.
The forward orbit from (1.4, 1.9) experienced a failure at (-5.2e-14, 0.055).
Problem is singular or tolerances are too large.
The backward orbit from (1.4, 1.9) left the computation window.
Ready.