Homework 9

Due: Tuesday, November 17.

$**$ 1) (practice with the argument that shows $dV/dt = \int_{D(t)} \nabla\cdot\mathbf{u}\,d\mathbf{x}$) Consider a fluid (gas or liquid) moving in three-dimensional space. Its velocity $\mathbf{u}$ is a function of position and time:

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \ ,$$

and so is its density ρ:

$$\rho = \rho(\mathbf{x}, t) \ .$$

(Density = mass per unit volume.) Let's look again at a region $D(t)$ moving with the flow. The total mass in $D(t)$ is

$$m(t) = \int_{D(t)} \rho(\mathbf{x}, t)\, d\mathbf{x} \ .$$

One of the fundamental equations of fluid dynamics is the *conservation of mass law*:

$$\frac{dm}{dt} = 0 \ .$$

This simply expresses that mass is not spontaneously created or lost. If you follow a region moving with the flow, the same particles are in that region for all time — that's what "moving with the flow" means!

a) Argue, as well as you can, that

$$0 = \frac{dm}{dt} = \int_{D(t)} \frac{\partial\rho}{\partial t}(\mathbf{x}, t) + \nabla\cdot(\rho\mathbf{u})(\mathbf{x}, t)\, d\mathbf{x} \ .$$

Hint: The mass in $D(t)$ changes due to two factors. First, the density changes as a function of time. Second, the boundary of $D(t)$ moves. These two correspond to the two summands in the integrand in the above equation.

b) Argue, as well as you can, that this implies

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \ .$$

Hint: The fact that the previous equation holds for *all* choices of $D(t)$ is the key here. If the integral of a function over *one* domain is zero, the function needn't be zero — for instance, $\int_0^{2\pi} \sin(x)\,dx$ is zero, but $\sin(x)$ is not zero. But if the integral of a *continuous* function over *all* domains is zero, then the function must indeed be zero — can you see why?)

This equation expresses the conservation of mass. There are similar equations expressing the conservation of momentum (three of them, since momentum is a vector with three components) and the conversation of energy (one, since energy is a scalar quantity). These five equations together are called the *Navier-Stokes equations*, and govern all fluid flow under normal conditions. Computer programs simulating fluid flow are based on the Navier-Stokes equations. The weather forecasts you see on TV are obtained by solving, on a computer, very similar equations to predict the flow of air in the atmosphere.

2) p. 342, problem 9.2.1, parts a) and b).

3) p. 343, problem 9.2.4.

4) p. 343, problem 9.2.6, part a).

5) p. 344, problem 9.3.8. (The *basin of attraction* of an attractor A is the set of all points $\mathbf{x_0}$ with the property that the trajectory $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x_0}$ converges to D, that is: $\lim_{t\to\infty} \mathbf{x}(t) \in A$.