Math 150, Fall 2009, Midterm Exam 2

The exam is due on Friday, November 20, *at the start of class*. When you hand in the exam, please keep this sheet. Please write your name on the paper you hand in, and *sign*. With your signature, you pledge that **you have not been in communication about this exam with anybody, neither with any other student in the class, nor with any other person, other than me**. Students found in violation of this pledge will receive an F in the course, and will be reported to the Dean of Students for further disciplinary action. Please note that this applies regardless of whether you receive or give help.

You are allowed to ask me questions, but I will answer only to clarify the meaning of a question.

You are allowed to use any material you like while working on this exam: the book, your notes, the internet, etc. Only communication with others is strictly prohibited.

Good luck!

1. A disease model. Suppose that $x = x(t)$ is the size of the healthy population at time t who are not immune, $y = y(t)$ the size of the sick population, and $z = z(t)$ the number of people who are immune against the disease. Assume that

$$\frac{dx}{dt} = -rxy, \tag{1}$$

$$\frac{dy}{dt} = rxy - sy, \tag{2}$$

$$\frac{dz}{dt} = sy, \tag{3}$$

for constants $r > 0$ and $s > 0$.

(a) Explain why Eqs. (1)–(3) make sense as a model of an infectious disease, assuming that every patient recovers and becomes immune to the disease.

Since Eqs. (1) and (2) don't contain z, we will now analyze just those two equations, omitting Eq. (3). We will focus exlusively on the quadrant $x \geq 0$ and $y \geq 0$.

(b) What are the fixed points of (1) and (2)?

(c) What are the nullclines of (1) and (2)?

(d) Verify that for any solution $(x(t), y(t))$ of (1) and (2) with $x > 0$ and $y > 0$,

$$y + x - \frac{s}{r} \ln x$$

is constant.

(e) Explain how you yourself could have discovered the conserved quantity in (d), by dividing Eq. (2) by Eq. (1), then expressing y as a function of x.

(f) Sketch the phase portrait for Eqs. (1) and (2).

(g) We say that an *epidemic* occurs if $dy/dt > 0$ initially. Under which condition does an epidemic occur? Explain why the conclusion is biologically reasonable.

(h) Explain: Trajectories that start in the quadrant $x > 0$, $y > 0$ never leave it.

2. Ruling out closed orbits using index theory. For the system

$$\begin{aligned} \frac{dx}{dt} &= 2x - 2x^2 - 5xy, \\ \frac{dy}{dt} &= y - y^2 - 2xy, \end{aligned}$$

use index theory to prove that there are no closed orbits.

3. Proving the existence of a closed orbit using the Poincaré-Bendixson Theorem. Show that the system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + y(4 - 2x^2 - 3y^2) \end{aligned}$$

has a closed orbit.

4. **A predator/prey model.** Consider the system

$$\begin{aligned} \dot{x} &= x\left[x(1-x) - y\right], \\ \dot{y} &= y(x-a). \end{aligned}$$

Assume $0 < a < 1$.

(a) This models a predator population, and a prey population. Which of the dependent variables x and y denotes the size of the predator population, and which the size of the prey population? Explain.

(b) We call the predator "modest" if $a < 1/2$, and "greedy" if $a > 1/2$. Explain what motivates those words.

(c) Find all fixed points. Show that there is one fixed point (x^*, y^*) with $x^* > 0$ and $y^* > 0$. Classify it. Your answer will depend on a.

(d) Explain: If $x(0) \geq 0$ and $y(0) \geq 0$, then $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \geq 0$. Trajectories do not leave the positive quadrant. We will assume $x \geq 0$ and $y \geq 0$ from here on.

(e) Show that a Hopf bifurcation occurs at $a = 1/2$.

(f) Show that for $a < 1/2$, there is a stable limit cycle. Hint: The key, of course, is to construct a trapping region. Let $(\xi(t), \eta(t))$ be the specific trajectory with

$$\xi(0) = 1, \quad \eta(0) = 1.$$

Show that there is a time $t^* > 0$ so that $\xi(t^*) = a$, and that $\xi(t)$ decreases and $\eta(t)$ increases for $0 \leq t \leq t^*$. Show that the region bounded by

$$x = 0, \quad y = 0, \quad x = 1, \quad (\xi(t), \eta(t)) \ (0 \leq t \leq t^*), \quad \text{and} \quad 0 \leq x \leq a, \ y = \eta(t^*)$$

is a trapping region.

(g) Sketch the phase portrait for a slightly below $1/2$, and for a slightly above $1/2$. (You may guess a little bit here. Classifying the fixed point at $x = 1$ and $y = 0$ will help.)

(h) Explain what the phase portraits tell you about the predator and prey populations when the predator is modest, and when the predator is greedy.