Solutions for the second midterm exam

1. (a) The rate at which people who are not immune get infected is proportional to x (the number of such people) and y (the number of sick people). That is Eq. (1). The number of sick people changes because health people get infected (this is the origin of the term rxy in Eq. (2)) and because sick people recover (this is the origin of the term $-sy$). People become immune by contracting the disease and recovering, that is Eq. (3). The model is known as the Kermack-McKendrick model. (Anderson Gray McKendrick (1876–1943), an epidemiologist in Schotland, was the first to use mathematics in the study of epidemics. I don't know who Kermack was.)

(b) At a fixed point, $x = 0$ or $y = 0$ by Eq. (1). But if $x = 0$ at a fixed point, then y has to be 0 anyway, by Eq. (2). So the fixed points are the point on the x-axis (all of them).

(c) The $dx/dt = 0$ nullcline consists of the x- and y-axes. The $dy/dt = 0$-nullcline consists of the x-axis, and the line $x = r/s$.

(d)

$$\frac{d}{dt}\left(y + x - \frac{s}{r}\ln x\right) = \dot{y} + \dot{x} - \frac{s}{r}\frac{\dot{x}}{x} = rxy - sy - rxy - \frac{s}{r}(-ry) = 0.$$

(e) Notice that as long as $x > 0$ and $y > 0$, $dx/dt < 0$. So x is a strictly decreasing function of t, and therefore y, a function of t, can also be viewed as a function of x. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

by the chain rule. So

$$\frac{dy}{dx} = \frac{rxy - sy}{-rxy} = \frac{s}{rx} - 1$$

and therefore

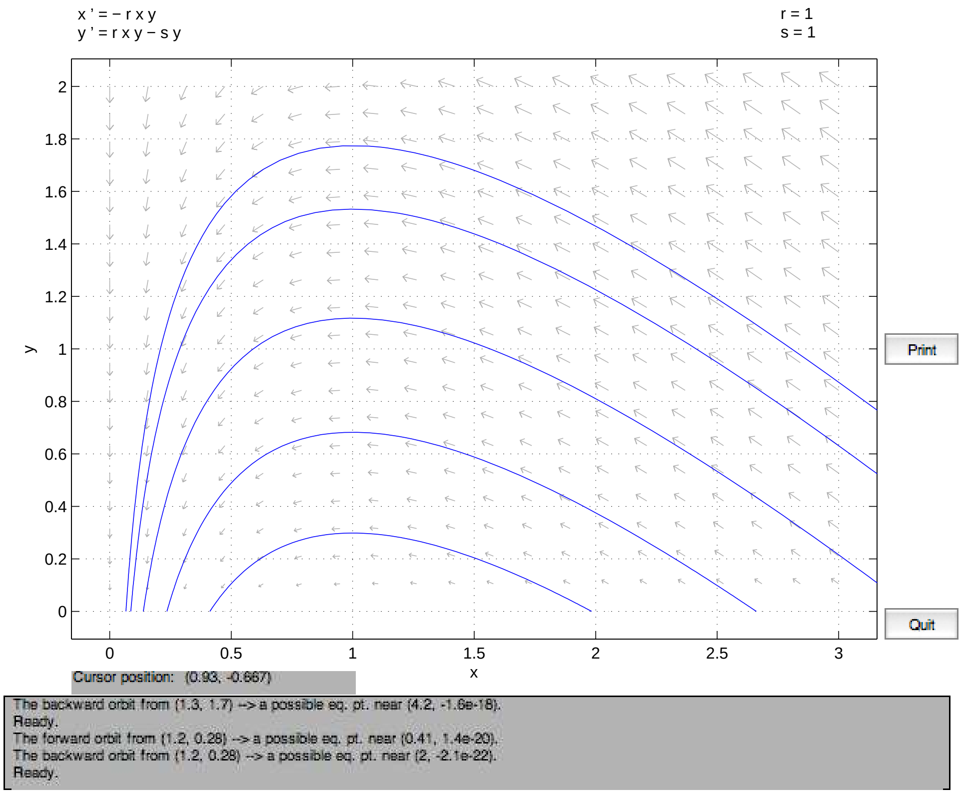
$$y = \frac{s}{r}\ln x - x + C.$$

This means precisely that

$$y + x - \frac{s}{r}\ln x$$

is a constant.

(f) I use Matlab:



You can easily see analytically that this is what the functions $y = (s/r)\ln x - x + C$ look like. The maximum is at $x = s/r$ in general. In my plot, $s/r = 1$.

(g) There is an epidemic if $x > s/r$, that is, if the population of healthy individuals that are not immune is sufficiently large. An epidemic occurs more easily if r is larger (that is, if the disease is more infectious) and if s is smaller (that is, if people take longer to recover). Both of these are intuitively unsurprising.

s/r is called the Kermack-McKendrick threshold. To see that this is really a dramatic threshold, think of the case $y = 1$, that is, there is initially one infected individual. When $x < s/r$, the total number of individuals who will contract the disease is going to be small, and independent of the size of the population. When $x < s/r$, it is going to be some percentage of the entire population.

(h) The x-axis consists of fixed points, and the positive y-axis is filled up with a fixed point (at $x = y = 0$) and a trajectory ($x(t) = 0$, $y(t) = e^{-st}$). So to leave the positive quadrant, a trajectory would have to cross other trajectories, which is impossible.

Note: It is not sufficient to say that $dx/dt = 0$ on the y-axis. That's true, but that doesn't mean that x could not go from positive to negative. For instance, $x(t) = t^3$ is positive for

$t > 0$, negative for $t < 0$, and nonetheless $dx/dt = 0$ for $t = 0$.

2. First think about the fixed points. (x,y) is a fixed point if and only if

$$\begin{aligned} 2x - 2x^2 - 5xy &= 0, \\ \text{and} \quad y - y^2 - 2xy &= 0. \end{aligned}$$

So (x,y) is a fixed point if and only if

$$x = 0 \quad \text{or} \quad 2 - 2x - 5y = 0$$

and

$$y = 0 \quad \text{or} \quad 1 - y - 2x = 0.$$

So the fixed points are

$$(0,0), \quad (0,1), \quad (1,0), \quad \text{and} \quad (0.375, 0.25).$$

Next, you have to find the Jacobi matrix:

$$J = \begin{bmatrix} 2 - 4x - 5y & -5x \\ -2y & 1 - 2y - 2x \end{bmatrix}.$$

Then plug in the fixed points:

$$(0,0): \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

So $(0,0)$ is an unstable node.

$$(1,0): \quad \begin{bmatrix} -2 & -5 \\ 0 & -1 \end{bmatrix}$$

So $\tau = -3$ and $\Delta = 2$. Since $\Delta < \tau^2/4$, $(1,0)$ is a stable node.

$$(0,1): \quad \begin{bmatrix} -3 & 0 \\ -2 & -1 \end{bmatrix}$$

So $\tau = -4$ and $\Delta = 3$. Since $\Delta < \tau^2/4$, $(0,1)$ is a stable node.

$$(0.375, 0.25): \quad \begin{bmatrix} -3/4 & -15/8 \\ -1/2 & -1/4 \end{bmatrix}$$

Here $\Delta = 3/16 - 15/16 < 0$, so $(0.375, 0.25)$ is a saddle.

Since the x-axis and the y-axis are filled up with trajectories of the system (this is what you have to say — to say they are nullclines is not enough, see solution to problem 1h), no trajectory can enclose the fixed points $(0,0)$, $(1,0)$, or $(0,1)$. So a closed orbit would either enclose no fixed point at all, or the fixed point $(0.375, 0.25)$ (but no others). Either way, the index could not be 1, as it has to be.

3. The only fixed point is at $(0,0)$. The Jacobi matrix at $(0,0)$ is

$$\begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}.$$

So $\tau = 4$ and $\Delta = 1$, therefore $(0,0)$ is an unstable node ($\Delta < \tau^2/4$). Now

$$\begin{bmatrix} y \\ -x + y(4 - 2x^2 - 3y^2) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = y^2(4 - 2x^2 - 3y^2).$$

If $x^2 + y^2 = R^2$, and R is sufficiently large ($R > \sqrt{2}$ is enough), then

$$4 - 2x^2 - 3y^2 \le 4 - 2(x^2 + y^2) = 4 - 2R^2 < 0.$$

Thus any solution crossing the circle $x^2 + y^2 = R^2$ must cross *into* the circle, not *out of the circle*.

Actually, there is a small subtlety here: What about trajectories escaping from the interior of the circle at one of the points $(R,0)$ or $(-R,0)$? At those points,

$$\begin{bmatrix} y \\ -x + y(4 - 2x^2 - 3y^2) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = y^2(4 - 2x^2 - 3y^2) = 0.$$

Could a trajectory escape the interior of the circle through those points? Suppose for instance a trajectory escaped the circle $x^2 + y^2 + R^2$, with $R > \sqrt{2}$, at $(R,0)$. At that point,

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 0 \\ -R \end{bmatrix}$$

points straight downwards. So a very short tie after crossing the circle $x^2 + y^2 = R^2$ at $(0,R)$, the trajectory would escape a slightly larger circle at a point (x,y) with $y < 0$, and it can't do that, as explained above. So indeed no trajectories can leave the interior of the circle $x^2 + y^2 = R^2$. If you remove a small disk centered at the origin from the disk with radius R, you get a trapping region without any fixed points, which by the Poincaré-Bendixson theorem must contain a closed orbit.

4. (a) x is the population size of the prey, y that of the predator.

(b) To grow in population size, the predator needs x to be greater than $1/2$. If a is small, the predator doesn't need much prey to grow in population size. If a is large, the predator needs a lot of prey to grow.

(c) (x,y) is a fixed point if

$$x[x(1-x)-y]=0$$

and

$$y(x-a)=0.$$

This means:

$$x=0 \quad \text{or} \quad y=x(1-x)$$

and

$$y=0 \quad \text{or} \quad x=a.$$

There are four possibilities:

$$x=0 \quad \text{and} \quad y=0$$

or

$$x=0 \quad \text{and} \quad x=a \quad \text{(but that's impossible of course!)}$$

or

$$y=x(1-x) \quad \text{and} \quad y=0$$

or

$$y=x(1-x) \quad \text{and} \quad x=a.$$

Thus there are three fixed points:

$$(0,0), \quad (1,0), \quad \text{and} \quad (a,a(1-a)).$$

Nobody survives, or the prey survives but the predator does not, or both survive. To classify the fixed points, we must compute the Jacobi matrix:

$$J=\begin{bmatrix} 2x-3x^2-y & -x \\ y & x-a \end{bmatrix}.$$

At $x=a$ and $y=a(1-a)$,

$$J=\begin{bmatrix} 2a-3a^2-a(1-a) & -a \\ a(1-a) & 0 \end{bmatrix}=\begin{bmatrix} a-2a^2 & -a \\ a-a^2 & 0 \end{bmatrix}.$$

This means $\tau=a-2a^2$ and $\Delta=a(a-a^2)$. Since $0<a<1$, $\Delta>0$, and $(a,a(1-a))$ is stable if $a-2a^2<0$, and unstable if $a-2a^2>0$. The inequality $a-2a^2<0$ means $a>1/2$. So the fixed point $(a,a(1-a))$ is stable if $a>1/2$ (the predator is greedy) and unstable if $a<1/2$ (the predator is modest).

(d) The x- and y-axes trap trajectories in the positive quadrant because they are filled with trajectories. (See solution to problem 1h. It is not enough to say that they are nullclines. Trajectories can cross nullclines, but they can't cross other trajectories.)

(e) The fixed point $(a, a(1-a))$ goes from a stable spiral to an unstable spiral as a falls below $1/2$.

(f) First we have to explain why the trajectory (ξ, η) referred to in the problem has the properties that I claimed it had. At $t = 0$, $(\xi, \eta) = (1, 1)$, and therefore $\dot{\xi} = -1$ and $\dot{\eta} = 1 - a > 0$. Therefore ξ decreases at least initially, and η increases. While ξ is greater than a, η keeps increasing, while ξ decreases, with

$$\dot{\xi} = \xi [\xi(1-\xi) - \eta] < -3/4a.$$

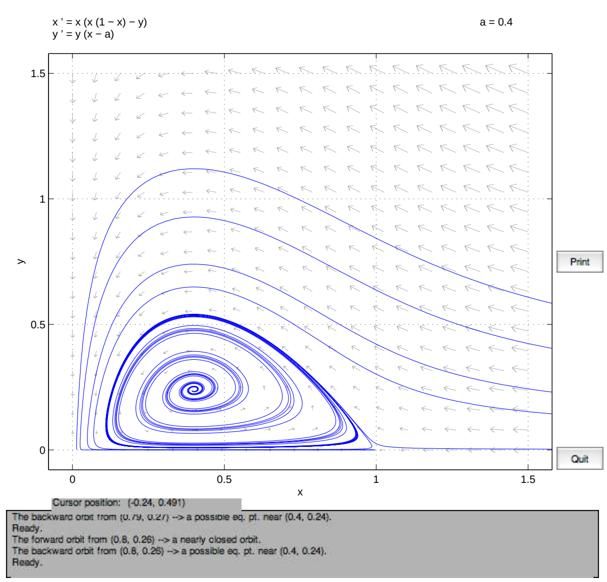
When ξ reaches a, then $\dot{\eta}$ becomes 0.

The trapping region that I proposed has five boundary segments, and we have to explain why none of those five segments can be crossed from the interior. For $x = 0$ and $y = 0$, that is simple: See part (d) of the problem. For $x = 1$, $\dot{x} = -y < 0$, so no trajectory can escape through the boundary segment $x = 1$. The trajectory (ξ, η) cannot be crossed by any other trajectory by the uniqueness theorem. Finally, the segment $0 \le x \le a$, $y = \eta(t^*)$ cannot be crossed by any trajectory because there

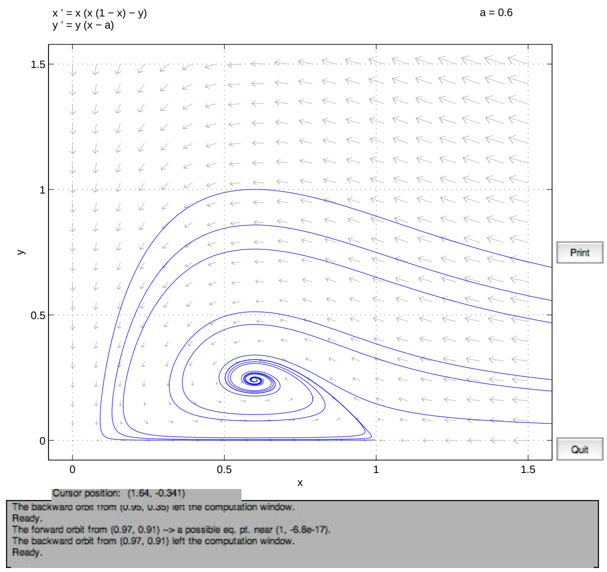
$$\dot{y} = y(x-a) < 0.$$

Therefore, if we cut out of the region that I proposed a small disk around the fixed point $(a, a(1-a))$, then we get, for $a < 1/2$, a fixed-point-free trapping region, which must contain a limit cycle.

(g) $a = 0.4$:



$a = 0.6$:



(h) There will be oscillations in the sizes of the two populations when the predator is modest. When the predator is greedy, the two populations will coexist at a stable fixed point. Notice that that fixed point is $(a, a(1-a))$: The greedier the predator, the *more* (!) prey will survive, and the *fewer* predators will survive!