Math 50, Fall 2009, Final Review Sheet, Solutions

The final exam for Math 50 will take place on Friday, December 18, from 3:30 to 5:30 p.m. in Anderson 206. To prepare the exam, do the review problems thoroughly, and leave it there.

1. (a) The linear approximation of $\sqrt{x}$ at $a = 100$ is

$$\sqrt{x} \approx 10 + \frac{x-100}{20}.$$

So

$$\sqrt{99} \approx 10 - \frac{1}{20} = 9.95.$$

(b) The quadratic approximation of $\sqrt{x}$ at $a = 100$ is

$$\sqrt{x} \approx 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000}.$$

This yields

$$\sqrt{99} \approx 9.95 - \frac{1}{8000} = 9.95 - 0.000125 = 9.949875.$$

(c) The cubic approximationis

$$\sqrt{x} \approx 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000} + \frac{(x-100)^3}{1,600,000}.$$

This yields

$$\sqrt{99} \approx 9.949875 - \frac{1}{1,600,000} = 9.949874375.$$

(These digits are all correct, except for the very last one.)

2.

$$\tan x \approx x + \frac{x^3}{3}$$

3. Here is the Taylor expansion as you know it well:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

You should think here of $x \approx a$. You could also write $x = a+h$, so $x - a = h$, and think of $h \approx 0$ then. With this notation, the formula becomes

$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f''''(a)}{4!}h^4 \ldots$$

Then, in this formula, you could also write "x" instead of "a" (it isn't the same "x" as before — now x is thought of as fix, and you vary h, whereas before, a was fixed, and you varied x):

$$f(x+h) \approx f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f''''(x)}{4!}h^4 + \dots$$

Or you could write "u" instead of "f":

$$u(x+h) \approx u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + \frac{u'''(x)}{3!}h^3 + \frac{u''''(x)}{4!}h^4 + \dots \quad (1)$$

Or, if you write "$-h$" instead of "h" in (1):

$$u(x-h) \approx u(x) - u'(x)h + \frac{u''(x)}{2}h^2 - \frac{u'''(x)}{3!}h^3 + \frac{u''''(x)}{4!}h^4 + \dots \quad (2)$$

The sum of (1) and (2) is

$$u(x+h) + u(x-h) \approx 2u(x) + u''(x)h^2 + \frac{u'''(x)}{12}u''''(x)h^4.$$

Subtract $2u(x)$ from both sides, then divide by h^2, to get the assertion.

4. [yet to be filled in]

5. (a) At height z, the cross-section of the paraboloid is a circle with a radius r, and z is proportional to r^2. (This is what "paraboloid" means.) So $z = Cr^2$ for some constant C. When $z = 4$, then $r = 4$, so $4 = C4^2$, or $C = 1/4$. Therefore

$$r = \sqrt{\frac{z}{C}} = 2\sqrt{z}.$$

Now consider a thin slice of the paraboloid, between heights z and $z + dz$. Its volume is approximately

$$\pi r^2 dz = \pi(2\sqrt{z})^2 dz = 4\pi z dz.$$

Its mass is therefore

$$\rho 4\pi z dz,$$

where ρ denotes the density of water, and its weight is

$$\rho g 4\pi z dz,$$

where g is the gravitational acceleration. To lift it out of the tank, it must be lifted a distance $4 - z$, so the work it takes to lift it out of the tank is

$$\rho g 4\pi z(4-z)dz.$$

Therefore the total work to empty the tank is

$$\int_0^4 \rho g 4\pi z(4-z)dz = \rho g\pi\left[8z^2 - \frac{4z^3}{3}\right]_0^4 = \rho g\pi\left[128 - \frac{256}{3}\right] = \rho g\pi\frac{128}{3}.$$

To evaluate this numerically, you have to remember that distance is measured in ft here. Therefore ρ should be measured in lbs/ft^3, and g in ft/sec^2. In these units, ρ is 62.4 (don't memorize that), and g is 32 (okay, maybe you want to memorize that one if you haven't yet, although $g \approx 9.81$m/sec^2 is of course how most of the world writes it).

(b) [yet to be filled in]

6. [problems 49 an 50 yet to be filled in]

problem 52. The side length of the square decreases linearly from b at height 0 to a at height h. If z denotes the height ($0 \le z \le h$), the side length of the square at height z is

$$L(z) = cz + d,$$

for some constants c and d. Since $L(0) = b$ and $L(h) = a$:

$$c0 + d = b, \quad ch + d = a.$$

This gives $d = b$, and then $ch + b = a$, or $c = (a-b)/h$. So

$$L(z) = \frac{a-b}{h}z + b.$$

Consider a small slice of the frustrum at height z and of thickness dz. Its volume is

$$L(z)^2 dz = \left[\frac{a-b}{h}z + b\right]^2 dz.$$

The volume of the frustrum is therefore

$$\int_0^h \left[\frac{a-b}{h}z + b\right]^2 dz = \frac{a^2 + ab + b^2}{3}h.$$

For $a = b$, this becomes $a^2 h$, as it should. For $a = 0$, it becomes

$$\frac{b^2}{3}h,$$

which is indeed the volume of a pyramid with base of size $b \times b$ and height h. (Don't memorize this formula. It's at least midly interesting to note, though, that volume-wise, exactly three pyramids of base $b \times b$ fit into a box of size $b \times b \times h$ — and now you'll probably remember the formula for the volume of a pyramid anyway!)

7. Let the temperature at time t (measured in hours past midnight) be $T(t)$. What we want to approximate is

$$\frac{1}{24}\int_0^{24} T(t)\,dt.$$

The given data allow us to approximate the integral using the trapezoid method. So the average temperature is about

$$\frac{1}{24}\left[6\times\frac{36+36}{2}+2\times\frac{36+32}{2}+4\times\frac{32+30}{2}+4\times\frac{30+28}{2}+4\times\frac{28+26}{2}+4\times\frac{26+22}{2}\right]$$

This is a weighted average of the seven measured temperatures. It can be written as

$$\frac{1}{8}\times 36+\frac{1}{6}\times 36+\frac{1}{8}\times 32+\frac{1}{6}\times 30+\frac{1}{6}\times 28+\frac{1}{6}\times 26+\frac{1}{12}\times 22\approx 30.333$$

(Note that the sum of the weights is $1/8+1/6+1/8+1/6+1/6+1/6+1/12=1$, as it should be.)

8. problem 1.

$$\int_0^5 \frac{x}{x+10}\,dx=\int_0^5 \frac{x+10-10}{x+10}\,dx=\int_0^5 1-\frac{10}{x+10}\,dx=\int_0^5 dx-\int_0^5 \frac{10}{x+10}\,dx=$$

$$5-10\,\ln(x+10)|_0^5=5-10\ln 15+10\ln 10=5+10\ln\frac{10}{15}=$$

$$5+10\ln\frac{2}{3}=5+10\,(\ln 2-\ln 3)\,.$$

problem 47.

$$\int_0^1 \frac{x-1}{\sqrt{x}}\,dx=\int_0^1 \sqrt{x}-\frac{1}{\sqrt{x}}\,dx=\int_0^1 x^{1/2}-x^{-1/2}dx=\left[\frac{x^{3/2}}{3/2}-\frac{x^{1/2}}{1/2}\right]_0^1=\frac{2}{3}-2=-\frac{4}{3}.$$

problem 49.

$$\int_{-\infty}^{\infty}\frac{dx}{4x^2+4x+5}=\int_{-\infty}^{\infty}\frac{dx}{(2x+1)^2+4}=\frac{1}{4}\int_{-\infty}^{\infty}\frac{dx}{(x+1/2)^2+1}=\ldots$$

(substitute $u=x+1/2$)

$$\frac{1}{4}\int_{-\infty}^{\infty}\frac{du}{1+u^2}=\frac{1}{4}\,\arctan(u)|_{-\infty}^{\infty}=\frac{\pi}{4}.$$

9.

problem 11. absolutely convergent, limit comparison test, compare with $1/n^2$

problem 12. divergent, limit comparison test, compare with $1/n$

problem 13. absolutely convergent, ratio test, the ratio converges to $1/5$, which is < 1 (and the terms are positive)

problem 14. conditionally convergent (alternating series test, without the alternating sign it is divergent: $p = 1/2$)

problem 15. divergent. The easiest way of seeing it is the integral test:

$$f(n) = \frac{1}{n\sqrt{\ln n}}$$

is positive, decreasing.

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}\, dx = \ldots$$

(substituting $u = \ln x$)

$$\ldots \int_{\ln 2}^\infty \frac{1}{\sqrt{u}}\, du = \infty.$$

problem 16. divergent. The n-th term does not converge to zero. It converges to $\ln(1/3)$ instead.

problem 17. absolutely convergent:

$$\left| \frac{\cos 3n}{1 + (1.2)^n} \right| \leq \frac{1}{(1.2)^n},$$

and

$$\sum_{n=1}^\infty \frac{1}{(1.2)^n} < \infty.$$

problem 18. absolutely convergent by the root test:

$$\sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \frac{n^2}{1+2n^2} \to \frac{1}{2} < 1.$$

problem 19. absolutely convergent by the ratio test. The ratio of the $(n+1)$-st term over the n-th term equals

$$\frac{2n+1}{5(n+1)},$$

which converges to $2/5 < 1$.

problem 20. divergent. The easiest way of seeing it is to use the ratio test. Don't be confused by the minus sign: $(-5)^{2n} = 25^n$. So this is actually a positive series, and the ratio of the $(n+1)$-st term over the n-th term converges to $25/9 > 1$.

problem 21. conditionally convergent. To show that it is convergent, using the alternating series test, you need to verify that

$$\frac{\sqrt{n}}{n+1}$$

is positive (obviously), tends to zero (obviously), and is decreasing. Here is why it is decreasing:

$$\frac{d}{dx}\frac{\sqrt{x}}{x+1} = \frac{(x+1)/(2\sqrt{x}) - \sqrt{x}}{(x+1)^2} = \dots$$

(assuming $x > 0$)

$$\dots \frac{-\sqrt{x}/2 + 1/(2\sqrt{x})}{(x+1)^2},$$

and this is clearly negative for large enough x (which is all that counts). To see that the series is not absolutely convergent, use the limit comparison test, comparing with $1/\sqrt{n}$.

problem 27. This is a geometric series, in essence:

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}} = -\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-3)^n}{8^n} = -\frac{1}{3}\sum_{n=1}^{\infty}(-3/8)^n = -\frac{1}{3}\frac{-3/8}{1+3/8} = -\frac{1}{3}\frac{-3}{8+3} = \frac{1}{11}$$

problem 30. To see what this is, it is best to write it out without a summation sign:

$$1 - \frac{\pi/9}{2!} + \frac{(\pi/9)^2}{4!} - \frac{(\pi/9)^3}{6!} + \dots$$

Now you must remember

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^6}{6!} + \dots = \cos x.$$

This is the Maclaurin expansion of $\cos x$. **You must remember the Maclaurin expansions of e^x, $\cos x$, $\sin x$, $\cosh x$, and $\sinh x$.** You find the ones for e^x, $\cos x$, and $\sin x$ on page 743. The one for $\cosh x$ is the same as for $\cos x$, but the minus signs are replaced by $+$-signs. Similarly the one for $\sinh x$ is the same as for $\sin x$, but the minus signs are replaced by $+$-signs.

The given series is therefore equal to $\cos(\pi/9)$.

problem 31. This is e^{-e}. To understand why, you must know the Maclaurin series of e^x.

10. problem 33.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \geq 1 + \frac{x^2}{2}.$$

problem 40. From the ratio test, this series converges if $|x| < 5$ and diverges if $|x| > 5$. The radius of convergence is therefore 5. To determine the interval of convergence, we have to look at $x = 5$ and $x = -5$. For $x = 5$, the series is

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2},$$

which is absolutely convergent. For $x = -5$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is absolutely convergent as well. So the interval of convergence is $[-5, 5]$ (the *closed* interval).

problem 42. By the ratio test, this is convergent for all real x. Therefore the radius of convergence is ∞, and interval of convergence is the real line $\mathbb{R}$.

problem 45. $f(a) = \sin(\pi/6) = 1/2$, $f'(a) = \cos(\pi/6) = \sqrt{3}2$, $f''(a) = -\sin(\pi/6) = -1/2$, $f''''(a) = -\cos(\pi/6) = -\sqrt{3}/2$, etc. So the Taylor expansion is

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2}\frac{(x-\pi/6)^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\pi/6)^3}{3!} + \frac{1}{2}\frac{(x-\pi/6)^4}{4!} + \frac{\sqrt{3}}{2}\frac{(x-\pi/6)^5}{5!} - \dots$$

11. To sketch the curve, notice that

$$t = x - 1,$$

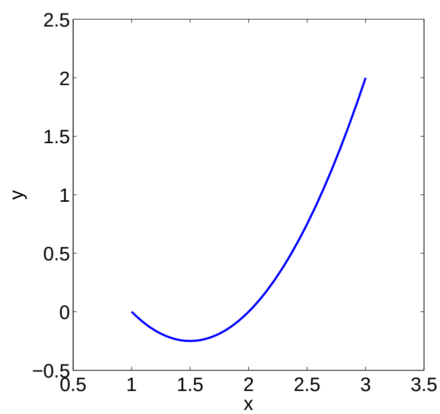
so

$$y = t^2 - t = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2.$$

This is a parabola that is easy to plot. The easiest way of plotting it is to complete the square:

$$y = x^2 - 3x + 2 = x^2 - 3x + \frac{9}{4} - \frac{1}{4} = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}.$$

So this is a parabola opening upwards, the minimum is at $x = 3/2$, and the minimum value is $-1/4$. The parameter t ranges from 0 to 2, so x ranges from 1 to 3:



The length of the parabola is

$$\int_0^2 \sqrt{1^2 + (2t-1)^2}\,dt$$

You should think of this as the integral from time $t = 0$ to time $t = 2$ of the *speed* of the moving point. To make this a little simpler, we can set $u = 2t - 1$, then the length is

$$\frac{1}{2}\int_{-1}^{3}\sqrt{1+u^2}\,du$$

12. Dividing by $-y^2$ on both sides of the equation, we find

$$-\frac{1}{y(t)^2}y'(t) = 1.$$

This should hold for all t. Writing "s" in place of "t", we find

$$-\frac{y'(s)}{y(s)^2} = 1$$

for all s. We integrate from $s = 1$ to $s = t$. (Because $y(1)$ is given, it is convenient to integrate from $s = 1$.) The result is

$$\int_1^t -\frac{y'(s)}{y(s)^2}\,ds = (t-1)+C.$$

This holds for all t. In particular, it holds for $t = 1$, and that gives $C = 0$. So

$$\int_1^t -\frac{y'(s)}{y(s)^2}\,ds = t-1.$$

Now substitute $u = y(s)$:

$$\int_{y(1)}^{y(t)} -\frac{1}{u^2}\,du = t-1.$$

Use $y(1) = 1$:

$$\int_{1}^{y(t)} -\frac{1}{u^2}\,du = t-1.$$

Evaluate the integral

$$\left[\frac{1}{u}\right]_1^{y(t)} = t-1.$$

So

$$\frac{1}{y(t)} - 1 = t-1,$$

or

$$y(t) = \frac{1}{t}.$$