Introduction to infinite series

"Infinite series" are infinite sums:

$$a_1 + a_2 + a_3 + a_4 + \ldots = \sum_{i=1}^{\infty} a_i = ?$$

When you cut it off after n terms, you get what is called the "n-th partial sum":

$$s_n = \sum_{i=1}^{n} a_i.$$

We say that the series "converges" if the limit of s_n as $n \to \infty$ exists, and in that case, we denote that limit by $\sum_{i=1}^{\infty} a_i$:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n \quad \text{if the limit exists.}$$

Otherwise we say that the series "diverges".

Why are we interested in infinite sums? Usually the interest in infinite sums is motivated by an interested in large, but finite sums. Large, finite sums are often well-approximated by infinite sums, and in many cases it is easier to understand and compute an infinite sum than a large, but finite one. You will see examples of this idea later on. Quite arguably the most important example are the infinite Taylor series:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots$$

If infinitely many terms are included on the right-hand side, is the series convergent? Often (but not always) the answer is "yes", and the sum of the series equals $f(x)$. If that is so, then we know that for large n,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(x)}{n!}(x-a)^n$$

with good accuracy. For x near a, $f(x)$ is approximately $f(a)$, but if that is not good enough for your purposes, then add $f'(a)(x-a)$ (that is the linear approximation). If that is still not good enough, then add $f''(a)/2(x-a)^2$ (that is the quadratic approximation). Note that $(x-a)^2$ is tiny if $x-a$ is small. And so on.

You will also hear more about infinite series in later courses when you learn about "Fourier series", the decomposition of a function into sinusoidal components — similar to the decomposition of a tone played on a violin into the fundamental pure tone and overtones.

If none of that motivates you, perhaps you can just find it interesting to think about infinite sums. For example, think about

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = \sum_{i=1}^{\infty} \frac{1}{2^i}.$$

Here

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8},$$

and so on. You can see easily that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

This is closely related to the ancient Greek paradox about the race between Achilles and a tortoise. The tortoise is allowed to start early. Then Achilles runs up to her, but by the time he gets there, the tortoise has advanced a bit. He runs up to the tortoise's new position, but by the time he gets there, the tortoise is again a little bit further along. And so on. He can never catch up! If we assume that Let's assume that Achilles is just twice as fast as the tortoise, and that he starts running when the tortoise is half a mile ahead. By the time he travels half a mile, the tortoise has traveled another quarter mile. By the time Achilles travels that quarter mile, the tortoise has traveled another eighth of the mile. And so on. Does it mean Achilles will never catch up? No, it means that he will catch up after traveling

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = 1$$

mile.

We will first think about the case when all summands in the infinite sum are positive:

$$a_i > 0 \quad \text{for all } i.$$

It turns out that understanding this case is more than half the battle. So think about

$$a_1 + a_2 + a_3 + ...$$

The sum can either come out to be finite, or infinite. That's all that can happen, since all terms are positive. It will be finite if the a_i become very small very fast as i increases, and infinite otherwise. It all comes down to understanding whether the a_i decrease fast enough as i increases.

A simple but very important special case:

$$f(1) + f(2) + f(3) + ...$$

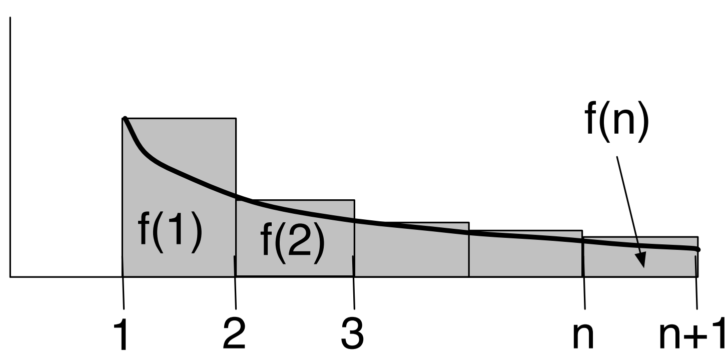
where $f = f(x)$ is a function defined for $x \geq 1$, with $f(x) > 0$ for all x, and f decreasing. The assumption that f be decreasing is crucial, as you will see in a moment. Note that

$$\sum_{i=1}^{\infty} \frac{1}{2^i}$$

is a special case: In this case, $f(x) = 1/2^x$. Many other interesting infinite series can be written in the form

$$\sum_{i=1}^{\infty} f(i)$$

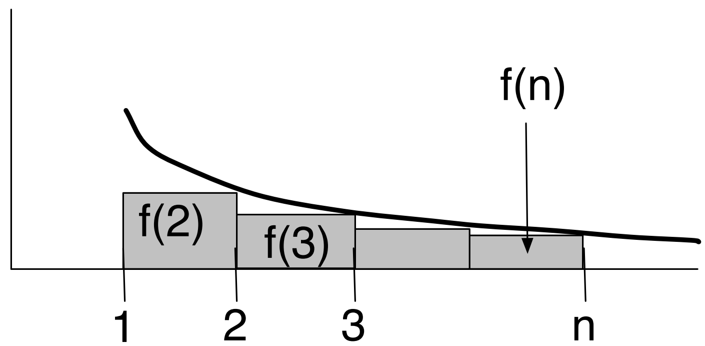
for a positive, decreasing function f. Look at this picture:



This picture shows:

$$\sum_{i=1}^{n} f(i) \geq \int_{1}^{n+1} f(x)\,dx. \tag{1}$$

Then look at this picture:



This picture shows:

$$\sum_{i=2}^{n} f(i) \leq \int_{1}^{n} f(x)\,dx.$$

We can add $f(1)$ to both sides of this inequality, and get:

$$\sum_{i=1}^{n} f(i) \leq \int_{1}^{n} f(x)\,dx + f(1). \tag{2}$$

Combining (1) and (2), we find:

$$\int_{1}^{n+1} f(x)\,dx \leq \sum_{i=1}^{n} f(i) \leq \int_{1}^{n} f(x)\,dx + f(1) \tag{3}$$

Taking the limit as $n \to \infty$:

$$\int_{1}^{\infty} f(x)\,dx \leq \sum_{i=1}^{\infty} f(i) \leq \int_{1}^{\infty} f(x)\,dx + f(1) \tag{4}$$

In particular, $\sum_{i=1}^{\infty} f(i) < \infty$ if and only if $\int_{1}^{\infty} f(x)dx < \infty$. This is called the *integral test*. It reduces the decision whether the infinite series converges to the decision whether the improper integral converges, and that is usually easier to determine.

Example 1:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

because

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1 < \infty.$$

The integral test is applicable because $f(x) = 1/x^2$ is positive and decreasing. More precisely, (4) tells us that the sum lies between 1 and $1 + f(1) = 2$. In fact, it is known (but will not be proved here) that

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \approx 1.6449.$$

Example 2:

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

because

$$\int_1^{\infty} \frac{1}{x}\,dx = \infty.$$

The integral test is applicable because $f(x) = 1/x$ is positive and decreasing. This is called the "harmonic series".

Example 3:

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} = \infty$$

because

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \infty.$$

The integral test is applicable because $f(x) = 1/\sqrt{x}$ is positive and decreasing.

Example 4: (generalization of Examples 1 through 3)

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \begin{cases} = \infty & \text{if } p \leq 1, \\ < \infty & \text{if } p > 1. \end{cases}$$

So $1/i^p$ decays to zero fast enough for a finite sum if $p > 1$, not if $p \leq 1$.

Example 5:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

as we said earlier. We could apply the integral test here, since

$$f(x) = \frac{1}{2^x}$$

is a positive, decreasing function.

$$\int_1^{\infty} \frac{1}{2^x}\, dx = \int_1^{\infty} 2^{-x}\, dx = \int_1^{\infty} e^{-x\ln 2}\, dx =$$

$$-\frac{1}{\ln 2} e^{-x\ln 2}\Big|_1^{\infty} = \frac{e^{-\ln 2}}{\ln 2} = \frac{1}{2\ln 2}.$$

So this implies that

$$\sum_{i=1}^{\infty} \frac{1}{2^i}$$

is finite and lies between $1/(2\ln 2) \approx 0.72$ and $1/(2\ln 2) + 1/2 \approx 1.22$. Nothing new there — we knew that the sum exists and equals 1.

Example 6:

$$\sum_{i=1}^{\infty} \frac{2i+3}{i+4} = \infty$$

because the summands

$$a_i = \frac{2i+3}{i+4}$$

don't even converge to zero as $i \to \infty$, they converge to 2! For convergence, they have to converge to zero *fast enough* — not only don't they do it fast enough here, they don't do it at all. For large i, the ith summand is about 2.

What about infinite series where some of the summands are negative? If the summands are *all* negative, as in

$$\sum_{i=1}^{\infty}\left(-\frac{1}{i^2}\right),$$

then its' simple: Of course that's just

$$-\sum_{i=1}^{\infty}\frac{1}{i^2}.$$

If only *finitely many* summands are negative, then it's simple too, as in the silly example

$$-\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

Of course that is convergent. For the decision whether or not an infinite series converges, finitely many terms never matter. The sum is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots-2\times\left(\frac{1}{2}+\frac{1}{4}\right)=1-2\times\frac{3}{4}=-\frac{1}{2}.$$

If only finitely many summands are positive, we multiply by -1 and are back to the case where only finitely many summands are negative.

So the only interesting case that is different from the case when all summands are positive is the case when infinitely summands are positive, and infinitely many are negative. In this case, we simply check whether or not

$$\sum_{i=1}^{\infty}|a_i|,$$

which is a sum of positive terms, is finite. Here are the possible cases:

CASE 1:

$$\sum_{i=1}^{\infty}|a_i|<\infty.$$

Then

$$\sum_{i=1}^{\infty}a_i$$

is convergent. One calls it “absolutely convergent” in this case.

<u>Example 7:</u>

$$\sum_{i=1}^{\infty} \frac{\sin(i)}{i^2}$$

is absolutely convergent because

$$\sum_{i=1}^{\infty} \left| \frac{\sin(i)}{i^2} \right| \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

(We used here that $|\sin(i)| \leq 1$ no matter what i is.)

CASE II:

$$\sum_{i=1}^{\infty} |a_i| = \infty.$$

In this case there are two possibilities:

CASE IIa: The series $\sum_{i=1}^{\infty} a_i$ is divergent.

CASE IIb: The series $\sum_{i=1}^{\infty} a_i$ is convergent. One calls it “conditionally convergent” in this case. Conditionally convergent infinite series have a peculiar property: The sum can change when the summands are re-ordered. In fact, something much stronger is true: If you give me any limit L that you would like achieve (even $L = \infty$ or $L = -\infty$ are allowed here), I can re-order the summands in such a way that the sum becomes L. So the commutative law fails in the most spectacular way for conditionally convergent series. The value of the sum depends on how the summands are ordered — hence “conditionally” convergent.

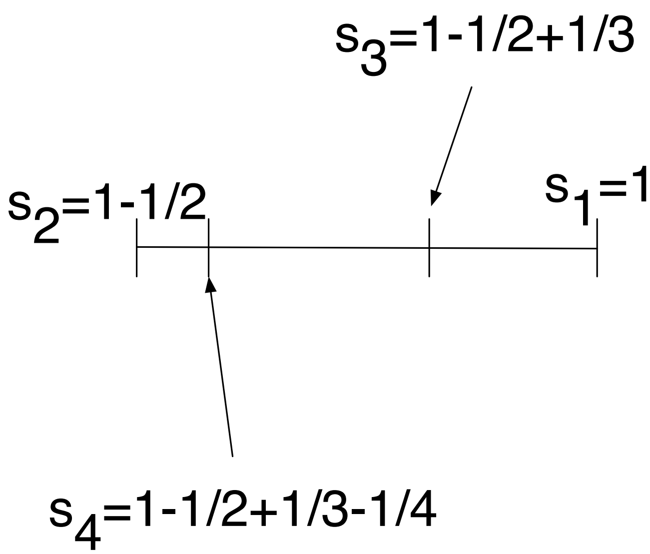
Case IIb is a strange borderline case: You do get convergence, but really just barely — the commutative law is not valid. It is quite arguably not very important. So there are two possibilities: Either $\sum_{i=1}^{\infty} |a_i|$ converges, and then so does

$\sum_{i=1}^{\infty} a_i$. Or $\sum_{i=1}^{\infty} |a_i|$ diverges, and then so does $\sum_{i=1}^{\infty} a_i$, unless it converges in a very weak sense, namely conditionally.

Example 8:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

is not absolutely convergent: If you take the absolute values, you get the harmonic series, which is not convergent. It is, however, convergent (thus conditionally convergent), as the following picture demonstrates.



The partial sums bounce back and fourth, with smaller and smaller jumps, and eventually settle on some value between 1/2 and 1. That value is known to be $\ln 2 \approx 0.693$, by the way, but we won't prove that here. This is called the "alternating harmonic series".

Let me explain how I could re-order this series to make the sum come out to be 42. I first add a bunch of positive terms:

$$1 + 1/3 + 1/5 + 1/7 + \ldots$$

I keep going until the sum exceeds 42. This will happen eventually, since the infinite sum

$$1 + 1/3 + 1/5 + 1/7 + 1/9 + 1/11 + \ldots$$

is divergent. As soon as it happens, I add in the first negative term, which is $-1/2$. This will bring the sum back down below 42. Then I add in positive terms that I had not added in yet, until the sum goes back up above 42. By now, it will go only just barely above 42, since all the positive terms left over are small. But their sum is still infinite, so I can in fact add enough to bring the sum above 42 again. Then I add in the next negative term, which is $-1/4$. And so on. As you see, I keep overshooting, then undershooting, then overshooting, etc. But the distance by which I overshoot and undershoot keeps getting smaller. So the sum will indeed become 42.

Of course, nothing was special with 42. I could do this with any limit L that you like.

Homework: 1. Using the integral test, prove that

$$\sum_{i=1}^{\infty} \frac{1}{3^i}$$

converges. 2. Using the integral test, prove that

$$\sum_{i=1}^{\infty} r^i$$

converges if $0 < r < 1$. 3. Deduce from the result of problem 2 that

$$\sum_{i=1}^{\infty} r^i$$

converges if $|r| < 1$. (So r can lie between -1 and 0 as well, and it still converges.) 4. Does

$$\sum_{i=1}^{\infty} \frac{1}{\cosh(i)}$$

converge? 5. Does

$$\sum_{i=1}^{\infty} \sin\left(\frac{1}{i}\right)$$

converge? (Hint: First show that $\sin(x) \geq \frac{2}{\pi}x$ for $0 \leq x \leq \pi/2$. If you don't see why, plot $\sin(x)$ and $\frac{2}{\pi}x$ in a single graph.) 6. Explain rigorously why

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+... = \infty.$$

7. Explain rigorously why

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+... = \infty.$$

8. Does

$$\sum_{i=2}^{\infty}\frac{(-1)^i}{2i\ln i}$$

converge? If so, does it converge absolutely or conditionally?