Due: Tuesday, December 8

1. The four-leaved rose is given by

$$r = \cos(2\theta), \quad 0 \le \theta \le 2\pi.$$

Express the length of this curve as an integral. (Do not evaluate the integral. It cannot be evaluated by finding an anti-derivative. Of course it could be evaluated using, for instance, the trapezoid method.)

2. Let us return to the curve given by

$$x(t) = t^2, \quad y(t) = t^3, \quad -1 \le t \le 1.$$

a) Think of this as the path of an ant walking in the (x, y)-plane. Explain why the speed of the ant at time $t = 0$ is zero. (This is why it is possible for there to be a cusp in the path at time $t = 0$.)

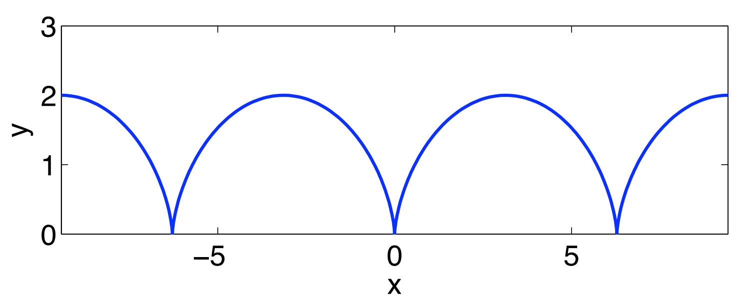
b) Express the length of the curve as an integral. (Notice that I restricted t to the interval from -1 to 1.)

c) Evaluate the integral.

3. The cycloid is given by

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad -\infty < \theta < \infty.$$

(I have taken the radius of the wheel to be 1 here.) It looks like this:



a) Verify that

$$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$$

when $\theta = 2\pi k$, k integer. (This is why it is possible for there to be cusps in the curve at $\theta = 2\pi k$.)

b) Now think of y as a function of x. Show that $dy/dx \to \infty$ as $x \to 2\pi k$ from the right, and $dy/dx \to -\infty$ as $x \to 2\pi k$ from the left (k integer). Hint:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Thus the sharp corners in the curve are in fact cusps — the angles are zero.

4. Here is how we analyzed the equation $dy/dt = ry$ in class. We want to find functions $y = y(t)$ with

$$y'(t) = ry(t) \quad \text{for all } t. \tag{1}$$

Here r is a number that we assume to be given. Let us assume that $y(t) > 0$ for all t, for argument's sake. We write (1) equivalently as follows:

$$\frac{y'(t)}{y(t)} = r \quad \text{for all } t. \tag{2}$$

Instead of "t", let us now use the letter "s", so that the letter "t" can later be used for a different purpose:

$$\frac{y'(s)}{y(s)} = r \quad \text{for all } s. \tag{3}$$

Eq. (3) can equivalently be written as

$$\int_0^t \frac{y'(s)}{y(s)}\, ds = \int_0^t r\, ds \quad \text{for all } t. \tag{4}$$

On the left-hand side of Eq. (4), we make the substitution $u = y(s)$, and we evaluate the right-hand side of Eq. (4):

$$\int_{y(0)}^{y(t)} \frac{1}{u}\, du = rt \quad \text{for all } t. \tag{5}$$

Now we integrate, remembering that we are assuming $y(t) > 0$ for all t:

$$\ln y(t) - \ln y(0) = rt \quad \text{for all } t. \tag{6}$$

We add $\ln y(0)$ to both sides of the equation:

$$\ln y(t) = \ln y(0) + rt \quad \text{for all } t. \tag{7}$$

We exonentiate both sides of the equation:

$$y(t) = y(0)e^{rt} \quad \text{for all } t. \tag{8}$$

Thus we have concluded that (8) is equivalent to (1), so (8) describes all positive solutions of the differential equation (1). (In fact, it describes all solutions, positive or negative, of (1), but I assumed $y(t) > 0$ for all t here to make the argument a little simpler.)

a) It is clear that (3) implies (4): If (3) holds, then so does (4). But why are (3) and (4) equivalent? So if (4) holds, how do you know that then also (3) must hold? Explain how you know that.

b) Repeat the precisely analogous sequence of arguments for the differential equation

$$y'(t) = y(t)^2 \quad \text{for all } t.$$

($y(t)^2$ denotes the square of $y(t)$.)