Math 50, Fall 2009, review for first midterm exam

The exam will take place on Friday, October 16, in our usual classroom. You will not be allowed to use notes, books, or calculators.

1. Suppose that you know $g(2) = -3$ and $g'(x) = \sqrt{x^2+12}$ for all x (a) Use a linear approximation to estimate $g(1.95)$. (b) Is your estimate too large or too small? Explain.

2. Suppose that $g(1) = 0$ and $g'(x) = \ln x$ for $x > 0$. Use a quadratic Taylor polynomial to estimate $g(1.1)$.

3. Find the linear and quadratic approximations of (a) $\tan x$ at $a = 0$, (b) $\tanh x$ at $a = 0$, (c) $\sqrt{x}$ at $a = 1$.

4. (a) State the definition of $\tanh(x)$. (b) Sketch the graph of $\tanh(x)$. Indicate horizontal and/or vertical asymptotes, if any. (c) Sketch the graph of $x = \operatorname{arctanh}(y)$, the inverse of $y = \tanh(x)$. Indicate horizontal and/or vertical asymptotes, if any. (d) The derivative of $\operatorname{acrtanh}(y)$ is $1/(1-y^2)$. Explain why.

5. (a) Use an integral to give an approximation to

$$\sum_{i=1}^{20} i^5$$

that is (slightly) too small. (b) Use an integral to give an approximation to the sum in part (a) that is (slightly) too large.

6. Express the limit as a definite integral:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}.$$

7. State the first part of the fundamental theorem of calculus.

8. Calculate the following integrals:

$$(a)\ \int_0^{\pi} x\sin x\,dx \qquad (b)\ \int xe^{-4x^2}\,dx \qquad (c)\ \int x\ln x\,dx$$

$$(d)\ \int_e^{e^2} \frac{dx}{x\ln x} \qquad (e)\ \int \frac{\sin x}{\sqrt{1+\cos x}}\,dx \qquad (f)\ \int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

(Hint for part (c): Remember that $\ln x^2 = 2\ln x$. Hint for part (f):

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = \int_0^1 \frac{x^3+x}{\sqrt{x^2+1}}\,dx - \int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx =$$

$$\int_0^1 x\frac{x^2+1}{\sqrt{x^2+1}}\,dx - \int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$$

Now these two integrals can be done by substitution.)

9. Evaluate

$$\int_0^1 \sqrt{1-x^2}\,dx$$

by interpreting it as an area. (Hint: $y=\sqrt{1-x^2}$ implies $y^2=1-x^2$, or $x^2+y^2=1$.)

10. The following function plays a role in the theory of diffraction of light waves:

$$C(x)=\int_0^x \cos\left(\frac{\pi t^2}{2}\right)\,dt.$$

(a) On which intervals is $C(x)$ increasing? (b) On which intervals is C concave up?

11. There is a serious water leak in your basement. Every day at noon, you measure the rate at which water is flowing out. Here is what you find:

Monday at noon: 0.1 gallons per hour
Tuesday at noon: 0.1 gallons per hour
Wednesday at noon: 0.2 gallons per hour
Thursday at noon: 0.3 gallons per hour
Friday at noon: 0.4 gallons per hour

Using a midpoint Riemann sum, estimate the total amount of water that leaks between Monday morning at midnight and Friday evening at midnight.

12. (a) Plot the region in the plane enclosed by the curves

$$y=\sin x \quad \text{and} \quad y=e^x$$

between $x=0$ and $x=\pi$. (b) Compute the area of the region you plotted in part (a).

13. The base of a cone is a circle with radius r. The height of the cone is h. Express the volume of the cone as an integral, and evaluate the integral.

14. (a) Using substitution, show:

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \cos^2 x\,dx.$$

(b) From (a), derive:

$$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}.$$