# Math 50, Fall 2009, review for first midterm exam, solutions

1. (a) $g(1.95) \approx g(2) - g'(2) \times 0.05 = -3 - 4 \times 0.05 = -3.2$ (b) $g'(x)$ is an increasing function, so the graph of $g$ is concave-up, so the linear approximation is too small.

2. $g''(x) = 1/x$, so $g(1) = 0$, $g'(1) = \ln 1 = 0$, and $g''(1) = 1/1 = 1$. Therefore $g(1.1) \approx g(1) + g'(1) \times 0.1 + g''(1)/2 \times 0.1^2 = 0.005$.

3. (a) You need to know here that $\tan(0) = 0$, $\tan'(x) = 1/\cos^2(x)$ and therefore $\tan'(0) = 1$, and $\tan''(x) = 2\cos^{-3}(x)\sin x$ and therefore $\tan''(0) = 0$. So the linear and quadratic approximations are the same: Both are $\tan(x) \approx x$. (b) You need to know here that $\tanh(0) = 0$, $\tanh'(x) = 1/\cosh^2(x)$ and therefore $\tanh'(0) = 1$, and $\tanh''(x) = -2\cosh^{-3}(x)\sinh(x)$ and therefore $\tanh''(0) = 0$. So the linear and quadratic approximations are the same: Both are $\tanh(x) \approx x$. (c) Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = (1/2)x^{-1/2}$, and $f''(x) = -(1/4)x^{-3/2}$. Therefore $f(1) = 1$, $f'(1) = 1/2$, and $f''(1) = -1/4$. The quadratic approximation at $a = 1$ is $1 + (1/2)(x-1) - (1/8)(x-1)^2 = -x^2/8 + 3x/4 + 3/8$. (You need not do the last step. The answer $1 + (1/2)(x-1) - 1/8(x-1)^2$ is sufficient, and in fact preferable, since it shows precisely how $\sqrt{x}$ differs from 1 when $x \approx 1$.)

4. (a) $\tanh(x) = \sinh(x)/\cosh(x) = (e^x - e^{-x})/(e^x + e^{-x})$ (b) Figure 3 on page 255. (c) Figure 10 on page 257. In this context, I like to call the horizontal axis the "$y$-axis", and the vertical axis the "$x$-axis". The book does it the conventional way — that is, the other way around. Either is fine. (d)

$$\frac{d}{dy}\text{arctanh}(y) = \frac{1}{\tanh'(x)} = \cosh^2(x),$$

where $\tanh(x) = y$. We now want to write $\cosh^2(x)$ in terms of $y$, and we don't know immediately how to do that. However, $\tanh(x) = y$ means

$$\frac{\sinh(x)}{\cosh(x)} = y,$$

so

$$\sinh(x) = y\cosh(x),$$

and squaring both sides we get

$$\sinh^2(x) = y^2\cosh^2(x).$$

Now using $\cosh^2 - \sinh^2 = 1$, we find

$$\cosh^2(x) - 1 = y^2\cosh^2(x),$$

or

$$\cosh^2(x) = \frac{1}{1-y^2}.$$

5. (a)

$$\sum_{i=1}^{20} i^5$$

is the right Riemann sum for

$$\int_0^{20} x^5\,dx$$

with $\Delta x = 1$. Since $x^5$ is increasing, the right Riemann sum is greater than the integral. So we find:

$$\sum_{i=1}^{20} i^5 > \int_0^{20} x^5\,dx = \frac{20^6}{6} = \frac{2^6}{6} \times 10^6$$

$$= \frac{64}{6} \times 10^6 = \frac{32}{3} \times 10^6 = 10.666... \times 10^6 = 1.0666... \times 10^7.$$

(b)

$$\sum_{i=1}^{20} i^5$$

is the left Riemann sum for

$$\int_1^{21} x^5\,dx$$

with $\Delta x = 1$. Since $x^5$ is increasing, the left Riemann sum is smaller than the integral. So we find:

$$\sum_{i=1}^{20} i^5 < \int_1^{21} x^5\,dx = \frac{21^6}{6} - \frac{1}{6} \approx 1.429 \times 10^7.$$

(The last step is of course not one you are expected to do without a calculator.) So this all implies that

$$1.0666 \times 10^7 < \sum_{i=1}^{20} i^5 < 1.429 \times 10^7.$$

In fact, the sum equals $1.2333... \times 10^7$. (To compute the sum was not part of your problem, and I certainly would recommend using a programmable calculator or computer to do it.) Notice that this is very close to the average of the two estimates $1.0666 \times 10^7$ and $1.429 \times 10^7$. This has to do with the fact that the trapezoid approximation is better than the left or right Riemann sum. (If you want to understand why, ask me.)

6.

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(x)\big|_0^1 = \arctan(1) = \frac{\pi}{4}.$$

7. If $g = g(x)$ is a continuous function, then for any choice of $a$,

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

8. (a) Use integration by parts:

$$\int_0^\pi x\sin x\,dx = -x\cos x|_0^\pi + \int_0^\pi \cos x\,dx = \pi$$

(The integral $\int_0^\pi \cos x\,dx$ is zero by symmetry. Or of course you can calculate it using the Fundamental Theorem, and you get zero.)

(b) Use substitution: $u = -4x^2$, $du = -8xdx$:

$$\int xe^{-4x^2}\,dx = -\frac{1}{8}\int e^u\,du = -\frac{e^u}{8} + C = -\frac{e^{-4x^2}}{8} + C$$

(c) This is one of the two hardest ones (the other hard one is (f)):

$$\int x\ln x\,dx = \frac{1}{2}\int x\ln x^2\,dx.$$

Now use substitution: $s = x^2$, so $ds = 2xdx$:

$$\frac{1}{2}\int x\ln x^2\,dx = \frac{1}{4}\int \ln s\,ds$$

To compute $\int \ln s ds$, you use integration by parts, but there is a trick: Set $u = \ln s$, do $du = (1/s)ds$, and $dv = ds$, $v = s$:

$$\int \ln s ds = \int u dv = -\int v du + uv = -\int s(1/s)\,ds + s\ln s = -\int 1 ds + s\ln s = s\ln s - s + C$$

So altogether:

$$\int x\ln x dx = \frac{1}{4}(s\ln s - s) + C = \frac{1}{4}\left(x^2\ln x^2 - x^2\right) + C = \frac{2x^2\ln x - x^2}{4} + C\,.$$

(d) Use substitution $u = \ln x$, $du = (1/x)dx$:

$$\int_e^{e^2}\frac{dx}{x\ln x} = \int_1^2 \frac{1}{u}\,du = \ln(2) - \ln(1) = \ln(2).$$

(Notice that when you do substitution, you have to change the limits of integration: When $x = e$, then $u = \ln x = 1$, and when $x = e^2$, then $u = \ln x = 2$.)

(e) Use substitution: $u = 1+\cos x$, so $du = -\sin x$:

$$\int \frac{\sin x}{\sqrt{1+\cos x}}\,dx = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u}+C = -2\sqrt{1+\cos x}+C.$$

(f) Let's begin by computing the *indefinite* integral. There is a trick here, and if you don't see that trick, you may be stuck for a long time:

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{x^2+1}}\,xdx = \int \frac{x^2+1}{\sqrt{x^2+1}}\,xdx - \int \frac{1}{\sqrt{x^2+1}}\,xdx =$$

$$\int \sqrt{x^2+1}\,xdx - \int \frac{1}{\sqrt{x^2+1}}\,xdx$$

Now both integrals are done by the substitution $u = x^2+1$, $du = 2xdx$:

$$\frac{1}{2}\int u^{1/2}\,du - \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{3}u^{3/2} - u^{1/2} + C = \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C.$$

Now let's compute the definite integral:

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = \left[\frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2}\right]_0^1 = \frac{1}{3}2^{3/2} - \sqrt{2} - \frac{1}{3} + 1 =$$

$$\frac{2}{3}\sqrt{2} - \sqrt{2} + \frac{2}{3} = -\frac{1}{3}\sqrt{2} + \frac{2}{3}.$$

9. The area is half a circle of radius 1. So the integral is $\pi/2$.

10. (a) $C(x)$ is increasing if $C'(x)$ is positive.

$$C'(x) = \cos\left(\frac{\pi x^2}{2}\right)$$

is positive if and only if

$$0 < \frac{\pi x^2}{2} < \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} < \frac{\pi x^2}{2} < \frac{5\pi}{2} \quad \text{or} \quad \frac{7\pi}{2} < \frac{\pi x^2}{2} < \frac{9\pi}{2} \quad \text{or} \quad \ldots$$

So this means

$$|x| < 1 \quad \text{or} \quad \sqrt{3} < |x| < \sqrt{5} \quad \text{or} \quad \sqrt{7} < |x| < \sqrt{9} \quad \text{or} \quad \ldots$$

(b) The function $C$ is concave-up if and only if

$$C''(x) = -\pi x \sin\left(\frac{\pi x^2}{2}\right) > 0. \tag{1}$$

If you got just to this point, you are okay. To see when (1) is the case, distinguish the cases $x > 0$ and $x < 0$. For $x > 0$, $C''(x) > 0$ if and only if

$$k\pi < \frac{\pi x^2}{2} < (k+1)\pi$$

for an odd integer. This means

$$2k < x^2 < 2(k+1)$$

for an odd integer, or (remembering that we are assuming $x > 0$ right now)

$$\sqrt{2k} < x < \sqrt{2(k+1)}$$

for an odd integer. So

$$\sqrt{2} < x < \sqrt{4} \quad \text{or} \quad \sqrt{6} < x < \sqrt{8} \quad \text{or} \sqrt{10} < x < \sqrt{12} \quad \text{or} \quad \ldots$$

For $x < 0$, $C''(x) > 0$ if and only if

$$k\pi < \frac{\pi x^2}{2} < (k+1)\pi$$

for an even integer, so

$$2k < x^2 < 2(k+1)$$

for an even integer, or (remembering that we are assuming $x < 0$ now)

$$-\sqrt{2(k+1)} < x < -\sqrt{2k}$$

for an even integer, or

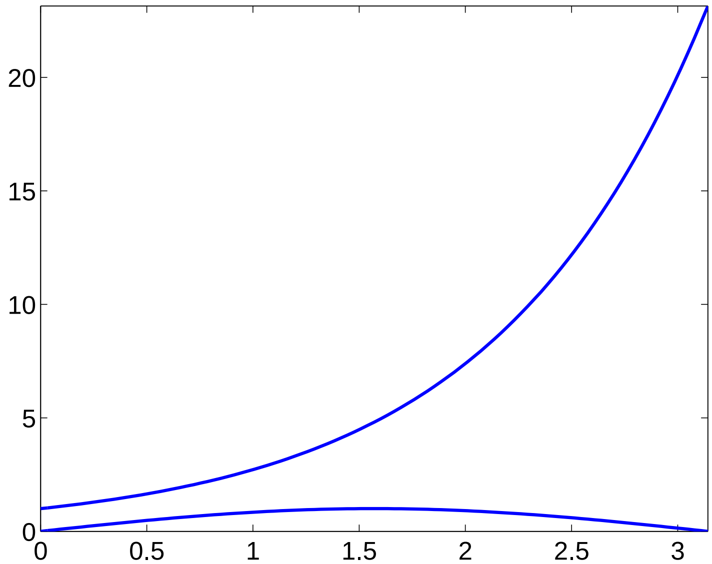
$$-\sqrt{2} < x < 0 \quad \text{or} \quad -\sqrt{6} < x < -\sqrt{4} \quad \text{or} -\sqrt{10} < x < -\sqrt{8} \quad \text{or} \quad \ldots$$

11. The total amount of water that leaks between Monday morning at midnight and Friday evening at midnight is about

$$(0.1+0.1+0.2+0.3+0.4)\times 24 = 1.1\times 24 = 24+2.4 = 26.4$$

gallons.

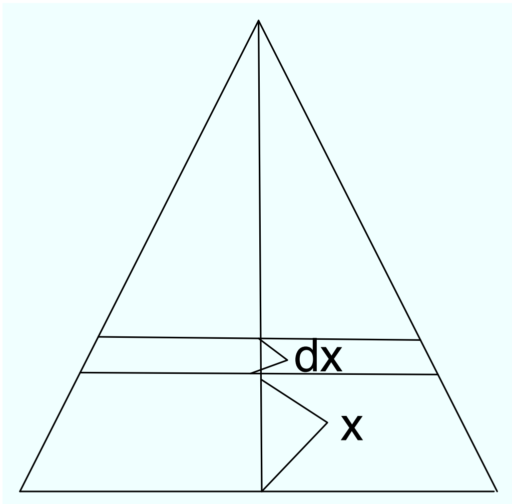
12. (a) Here are the two curves:




(b) The area is

$$\int_0^{\pi} (e^x - \sin x)\, dx = [e^x + \cos x]_0^{\pi} = e^{\pi} - 1 - 1 - 1 = e^{\pi} - 3.$$

13. Look at a slice of thickness $dx$ that is a distance $x$ from the ground:




This is a cylindrical slice, approximately, with thickness $dx$ and cross section equal to a circle with radius $r(1-x/h)$. (Why is this the radius? As $x$ increases, the radius must decrease linearly from $r$, when $x=0$, to 0, when $x=h$.) So the slice has volume

$$\pi r^2 \left(1 - \frac{x}{h}\right)^2 dx,$$

approximately. The total volume is therefore

$$\int_0^h \pi r^2 \left(1 - \frac{x}{h}\right)^2 dx = -\pi h r^2 \left[\frac{1}{3}\left(1 - \frac{x}{h}\right)^3\right]_0^h = \frac{\pi h r^2}{3}.$$

14. (a) Some of you saw that this is obvious by symmetry. That's a good argument, better than the one I am about to give in fact. But I want you to understand this argument as well:

$$\int_0^\pi \sin^2 x\, dx$$

can be reduced to an integral that involves cosine by noting that

$$\sin(x + \pi/2) = \cos x.$$

So let us write $x = u + \pi/2$, $dx = du$. Then

$$\int_0^\pi \sin^2 x\, dx = \int_{-\pi/2}^{\pi/2} \sin^2(u + \pi/2)\, du = \int_{-\pi/2}^{\pi/2} \cos^2(u)\, du.$$

Now make the observation that $\cos^2(u)$ is a periodic function with period $\pi$: $\cos^2(u + \pi) = \cos^2(u)$ for all $u$. Integrating over the interval from $-\pi/2$ to $\pi/2$ will therefore give the same as integrating over any other integral of length $\pi$, for example over the interval from 0 to $\pi$. So

$$\int_{-\pi/2}^{\pi/2} \cos^2(u)\, du = \int_0^\pi \cos^2(u)\, du.$$

(b) The sum of the two integrals is

$$\int_0^\pi (\sin^2 x + \cos^2 x) dx = \int_0^\pi 1 dx = \pi.$$

Since the two integrals are equal, by (a), each is $\pi/2$.