Math 50, fall 2009, review for second midterm exam, solutions

The exam will take place on Friday, November 13, in our usual classroom. You will not be allowed to use notes, books, or calculators.

1. (a) Substitute $u = \sin x$, $du = \cos x dx$:

$$\text{(a)} \quad \int \frac{\cos x}{\sin^x} dx = \int \frac{1}{u^2}\, du = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$$

You will be forgiven if you forget "$+C$", but you will *not* be forgiven if you don't convert back to x.

(b) Integration by parts with $u = x^2$ and $v = -e^{-x}$ gives:

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + \int 2xe^{-x}\, dx$$

Now integrate by parts again, with $u = 2x$ and $v = -e^{-x}$:

$$-x^2 e^{-x} + \int 2xe^{-x}\, dx = -x^2 e^{-x} - 2xe^{-x} + \int 2e^{-x}\, dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C =$$

$$e^{-x}(-x^2 - 2x - 2) + C$$

You should always simplify as far as possible, and write your answers as transparently as they can be written. (If you don't, you will often lose points.)

(c) Substitute $u = \arcsin(x)$:

$$\int_0^{1/2} \frac{\arcsin(x)}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} u\, du = \left.\frac{u^2}{2}\right|_0^{\pi/6} = \frac{\pi^2}{72}.$$

Note that here there is no need to convert back to x, since it's a definite integral. The answer is a number, not a function. You should know that $\sin(\pi/6) = 1/2$ (and therefore $\arcsin(1/2) = \pi/6$, and in general the values of sin, cos, and tan at $\pi/6$, $\pi/4$, $2\pi/6$, that is, at 30^o, 45^o, and 60^o.

(d) Substitute $u = t^{3/2}$:

$$\int \sqrt{t} \sin\left(1 + t^{3/2}\right)\, dt = \frac{2}{3} \sin(1+u)\, du = -\cos(1+u) + C = -\cos(1+t^{3/2}) + C$$

Again, if you don't write "$+C$" that's okay, I know that you know that. But if you don't convert back to t, then I'll have to assume that you don't understand that "$\int \sqrt{t} \sin\left(1 + t^{3/2}\right)\, dt$"

stands for "the anti-derivatives of $\sqrt{t}\sin\left(1+t^{3/2}\right)$", which of course are functions of t, and I'll have to deduct points of lack of understanding.

(e) The trick is to get rid of all sin-terms except one, using the Pythagorean theorem:

$$\int \sin^3 x \cos^2 x dx = \int (1-\cos^2 x)\, \cos^2 x\, \sin x\, dx.$$

Now you can substitute $u = \cos x$:

$$\int (1-\cos^2 x)\, \cos^2 x\, \sin x\, dx = -\int (1-u^2)u^2\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

This trick does not work if you have an integrand of the form $\sin^n x \cos^m x$ and both n and m are even. But then you can use double angle formulas instead — see (g) below.

(f) Use integration by parts with $u = \ln x$ and $v = x$:

$$\int \ln x\, dx = x \ln x - \int dx = x \ln x - x + C$$

(g)

$$\int \sin^2 u \cos^2 u\, du = \int (1-\cos^2 u) \cos^2 u\, du = \int \cos^2 u\, du - \int \cos^4 u\, du \qquad (1)$$

Now you use the double-angle formula

$$\cos^2 u = \frac{\cos(2u)+1}{2}. \qquad (2)$$

Are you expected to you know this? The short answer is yes. The long answer is: You don't have to memorize it if you instead memorize these three formulas:

$$\begin{aligned} \sin^2\alpha + \cos^2\alpha &= 1 & (3)\\ \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta & (4)\\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta & (5) \end{aligned}$$

These three you should memorize *for sure*. If you take $\beta = \alpha$, then (5) implies

$$\cos(2\alpha) = \cos^{\alpha} - \sin^2\alpha,$$

and by (3) that means

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$

or

$$\cos^2\alpha = \frac{\cos(2\alpha)+1}{2}.$$

If you trust your memory more than your algebra, memorize the double-angle formulas (you also need one for $\sin(2u)$). If, like me, you trust your algebra more than your memory, only memorize (3)–(5).

So now let's return to (1) and use (2):

$$\int \cos^2 u\,du - \int \cos^4 u\,du = \int \frac{\cos(2u)+1}{2} - \frac{(\cos(2u)+1)^2}{4}\,du$$

We do a little bit of algebra:

$$\frac{1}{4}\int 1-\cos^2(2u)\,du$$

Now use the double-angle formula once more:

$$\frac{1}{4}\int 1-\frac{\cos(4u)+1}{2}\,du = \frac{1}{4}\int \frac{1}{2}-\frac{\cos(4u)}{2}\,du = \frac{1}{4}\left[\frac{u}{2}-\frac{\sin(4u)}{8}\right]+C =$$

$$\frac{u}{8}-\frac{\sin(4u)}{32}+C$$

(h) This is the simplest case of a problem that you can solve using partial fraction decomposition. The denominator factors into two distinct linear factors:

$$x^2-3x+2=(x-2)(x-1).$$

In general, any function of the form

$$\frac{ax+b}{(x-x_1)(x-x_2)}$$

with a, b, x_1, x_2 constant and $x_1 \neq x_2$, can be written in the form

$$\frac{ax+b}{(x-x_1)(x-x_2)} = \frac{A}{x-x_1}+\frac{B}{x-x_2}\,.$$

In our example:

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1}\,.$$

To find A and B, multiply both sides by $x-2$ and also by $x-1$:

$$x = A(x-1)+B(x-2).$$

Simplify:

$$x = (A+B)x-(A+2B). \tag{6}$$

The right-hand side of Eq. (6) is a linear function. Since this linear function is in fact equal to x, its slope is 1:

$$A+B=1, \tag{7}$$

and its y-intercept is 0:

$$A+2B=0. \tag{8}$$

Eqs. (7) and (8) are two equations in the two unknowns A and B. It is easy to solve them, for example solve (7) for A: $A=1-B$, plug that into (8):

$$1-B+2B=0,$$

so $B=-1$, and then, from (7), $A=2$. So

$$\frac{x}{(x-2)(x-1)}=\frac{2}{x-2}-\frac{1}{x-1}.$$

Therefore

$$\int\frac{x}{x^2-3x+2}\,dx=\int\frac{x}{(x-2)(x-1)}\,dx=\int\frac{2}{x-2}\,dx-\int\frac{1}{x-1}\,dx=$$

$$2\ln|x-2|-\ln|x-1|+C=\ln\frac{(x-2)^2}{|x-1|}+C$$

This trick works in general for functions of the form

$$\frac{\text{linear (or constant) function}}{(x-x_1)(x-x_2)\text{ with }x_1\neq x_2}.$$

If the denominator is a quadratic function, but cannot be factored, then you complete the square, as in the next problem.

(i)

$$\int\frac{1}{2x^2+3}\,dx=?$$

The aim is to make the integrand look like

$$\frac{1}{u^2+1}$$

somehow. You first factor out $1/3$ to turn the "3" in the denominator into "1":

$$\int\frac{1}{2x^2+3}\,dx=\frac{1}{3}\int\frac{1}{(2/3)x^2+1}\,dx.$$

Now you set $u=\sqrt{2/3}\,x$:

$$\frac{1}{3}\sqrt{\frac{3}{2}}\int\frac{1}{u^2+1}du=\frac{1}{\sqrt{6}}\arctan(u)+C=\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}x\right)+C$$

(j) The integrand becomes infinite at 0, in the middle of the interval! Therefore the improper integral is convergent if and only if both

$$\lim_{t\to 0+} \int_t^1 \frac{1}{\sqrt{x}}\, dx$$

and

$$\lim_{t\to 0-} \int_{-1}^t \frac{1}{\sqrt{x}}\, dx$$

exist. In fact,

$$\lim_{t\to 0} \int_t^1 \frac{1}{\sqrt{x}}\, dx = \lim_{t\to 0}\ 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0} (2 - 2\sqrt{t}) = 2,$$

and similarly

$$\lim_{t\to 0-} \int_{-1}^t \frac{1}{\sqrt{x}}\, dx = 2.$$

So

$$\int_{-1}^1 \frac{1}{\sqrt{x}}\, dx$$

in fact is convergent, and its value is 4.

(k) Use integration by parts, with $u = x$ and $v = -e^{-x}$:

$$\int_0^\infty xe^{-x}\, dx = -\ xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 1.$$

(Strictly speaking you would have to write this with a limit as $t \to 0+$. Make sure you know how to.)

(l)

$$\int_{-\infty}^\infty x^2 e^{-x^2}\, dx$$

is a convergent improper integral if and only if both

$$\lim_{t\to\infty} \int_0^t x^2 e^{-x^2}\, dx$$

and

$$\lim_{t\to -\infty} \int_t^0 x^2 e^{-x^2}\, dx$$

exist. Let work on the first of these two limits:

$$\lim_{t\to\infty} \int_0^t x^2 e^{-x^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{x}{2} 2xe^{-x^2}\, dx$$

The reason why I wrote it this way is that now I can use integration by parts with $u = x/2$ and $v = -e^{-x^2}$:

$$\lim_{t\to\infty}\int_0^t \frac{x}{2}2xe^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\frac{x}{2}e^{-x^2}\right]_0^t + \frac{1}{2}\int_0^t e^{-x^2}\,dx\right) = \lim_{t\to\infty}\left(-te^{-t^2} + \frac{1}{2}\int_0^t e^{-x^2}\,dx\right) =$$

$$\frac{1}{2}\lim_{t\to\infty}\int_0^t e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

Now

$$\frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx$$

by symmetry, so

$$\frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$$

Altogether, we conclude that

$$\lim_{t\to\infty}\int_0^t x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{4}.$$

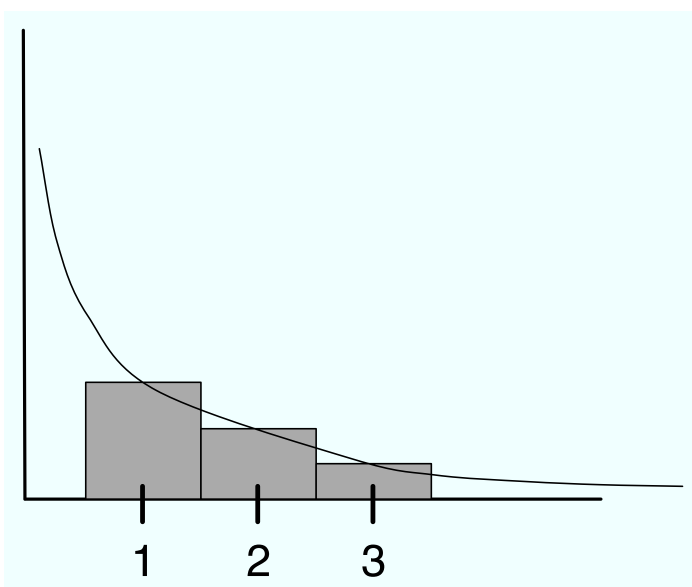
Analogously,

$$\lim_{t\to-\infty}\int_t^0 x^2e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}.$$

So

$$\int_{-\infty}^\infty x^2e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

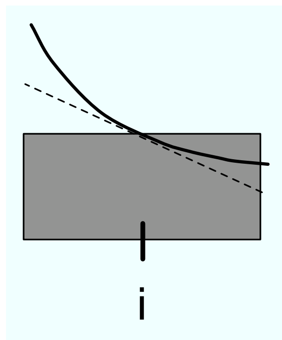
2. The picture is like this:



The sum of the areas of the rectangles is

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}.$$

It is smaller than the area under the curve from $x = 1/2$ to $x = n+1/2$. To see this, look at one of the rectangles:



The area of the rectangle equals the area under the dotted line (the tangent to the curve $1/\sqrt{x}$, which is smaller than the area under the curve. The only property of $1/\sqrt{x}$ that this argument uses is that it is concave-up, by the way.

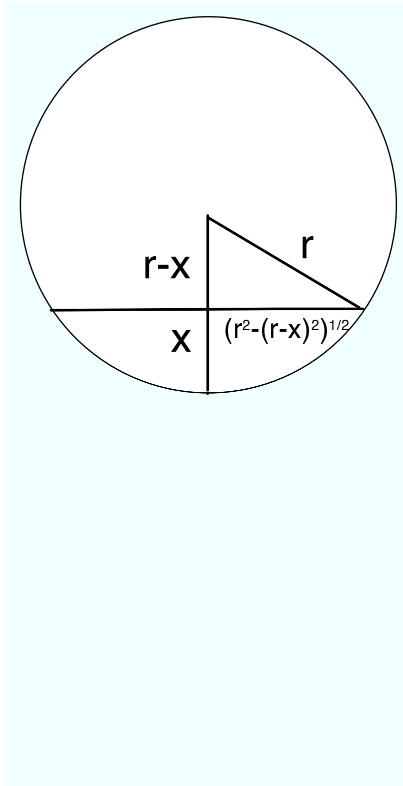
So

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} < \int_{1/2}^{n+1/2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{n+1/2} - 2\sqrt{1/2} = \sqrt{4n+2} - \sqrt{2}.$$

3. Look at a slice at height x above the south pole and thickness dx. Its cross-sectional area is

$$\pi\left[\sqrt{r^2-(r-x)^2}\right]^2 = \pi(r^2-(r-x)^2),$$

as you can see from the following picture:



So the volume of the slice is

$$\pi(r^2-(r-x)^2)dx,$$

and the volume of the cap of height h is

$$\int_0^h \pi(r^2-(r-x)^2)dx = \pi r^2 h + \pi\frac{(r-x)^3}{3}\bigg|_0^h = \pi r^2 h + \pi\frac{(r-h)^3}{3} - \pi\frac{r^3}{3} =$$

$$\frac{\pi}{3}\left(3r^2h + (r-h)^3 - r^3\right) = \frac{\pi}{3}h^2\left(3r-h\right)$$

4. page 722:

(1) Convergent positive series: Compare with $1/3^n$.

(3) Divergent: n-th term does not converge to zero.

(5) Absolutely convergent: Use the ratio test; the limit of the $n+1$-st term over the n-th term is $2/5$.

(7) Divergent:

$$\frac{1}{n\sqrt{\ln n}} > \frac{1}{n\ln n},$$

and even

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n} = \infty.$$

(We proved this in class using the integral test.)

9. Convergent positive series: Use the ratio test. The limit of the $n+1$-st term over the n-th term is $1/e$.

11. Not absolutely convergent (see problem (7)), but conditionally convergent by the alternating series test, since $1/(n \ln n)$ is positive, decreasing, converges to zero as $n \to \infty$. (You must state all three of these facts to get full credit.)

13. Convergent positive series: Use the ratio test. The limit of the $(n+1)$-st term over the n-th term is zero.

17. The n-th term does not converge to zero, so divergent.

19. Not absolutely convergent:

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty.$$

But convergent by the alternating series test, since $\ln n/\sqrt{n}$ is positive, decreasing for sufficiently larger n, and its limit is zero. To show that it is decreasing for sufficiently large n, define

$$f(x) = \frac{\ln x}{\sqrt{x}}$$

for $x \geq 1$ and take the derivative:

$$\frac{\sqrt{x}/x - \ln x/(2\sqrt{x})}{x} = \frac{1}{x^{3/2}} \left(1 - \frac{\ln x}{2}\right).$$

This is negative if $\ln x > 2$, that is, if $x > e^2$. To show that the limit of $f(x)$ as $x \to \infty$ is 0, use l'Hospital's Rule.

21. Absolutely convergent: Use the n-th root test. The n-th root of

$$\frac{2^{2n}}{n^n}$$

is

$$\frac{4}{n},$$

and this converges to zero. (Convergence to any number < 1 would be enough.)

23. Divergent positive series by the limit comparison test:

$$\tan(1/n) \sim 1/n$$

as $n \to \infty$.

25. Convergent positive series: ratio test. The ratio of the $n+1$-st term, divided by the n-th term, tends to zero. (Convergence to anything < 1 would be enough.)

27. Convergent positive series. You can use the limit comparison test:

$$\lim_{k\to\infty} \frac{\frac{k \ln k}{(k+1)^3}}{\frac{1}{k^{3/2}}} = 0,$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < \infty.$$

29. Absolutely convergent:

$$\frac{1}{\cosh(n)} < \frac{2}{e^n},$$

and

$$\sum_{n=1}^{\infty} \frac{2}{e^n} < \infty.$$

31. Does not converge since the k-th term does not tend to zero as $k \to \infty$. In fact,

$$\frac{5^k}{3^k + 4^k} = \frac{1}{(3/5)^k + (4/5)^k} \to \infty.$$

33. Convergent positive series: Use the limit comparison test, compare with $1/n^{3/2}$.

35. Convergent positive series. You can for instance use the n-th root test. The n-th root of

$$\left(\frac{n}{n+1}\right)^{n^2}$$

is

$$\left(\frac{n}{n+1}\right)^{n} = \frac{1}{(1+1/n)^n},$$

which tends to $1/e$ as $n \to \infty$.

37. Convergent positive series. To see it, note that

$$\sqrt[n]{2} - 1 = 2^{1/n} - 1 \to 0$$

as $n \to \infty$. Now use the n-th root test.

5. page 695, problem 58. (a) First it drops by H. Then it rises to rH, and drops by rH, then rises to r^2H and drops by r^2H, and so on. So the total distance traveled is

$$H + 2\sum_{k=1}^{\infty} r^k H = H + 2\frac{r}{1-r}H = \frac{1+r}{1-r}H.$$

You must know the formulas

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

and

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r},$$

both valid if $|r| < 1$. You need to memorize only one of these, since they very easily follow from each other.

(b) How long does it take to fall from height H to zero? The ball falls

$$\frac{1}{2}gt^2$$

meters in t seconds, so to calculate the time t needed to fall from height H, solve

$$\frac{1}{2}gt^2 = H :$$

$$t = \sqrt{\frac{2H}{g}}.$$

To fall from height rH takes, similarly, time

$$\sqrt{\frac{2rH}{g}}.$$

And so on. All the falls together take this time:

$$\sqrt{\frac{2H}{g}} \sum_{k=0}^{\infty} (\sqrt{r})^k = \sqrt{\frac{2H}{g}} \frac{1}{1-\sqrt{r}}.$$

Each fall is paired with a rise, immediately preceding it, that takes precisely equally long. So does this mean that the total travel time equals

$$2\sqrt{\frac{2H}{g}}\frac{1}{1-\sqrt{r}} \text{ ???}$$

Not quite: The very first fall, from height H to the ground, is not paired with a rise that takes the same amount of time. So the total travel time is actually

$$2\sqrt{\frac{2H}{g}}\frac{1}{1-\sqrt{r}} - \sqrt{\frac{2H}{g}} = \sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}}.$$

It is interesting that the total travel time is finite.