TUFTS UNIVERSITY
Department of Economics

MathCamp Fall 2005

Practice Problem Set 2 w/ solutions

0. Read on "Optimal Control" (for example, Chiang and Wainwright Chap 20, pp 630) [You will not be tested on optimal control]

1. (i) Solve the following problem:

$$\max_{x_1,x_2} f(x_1, x_2) = x_1x_2$$

subject to $x_1 + 4x_2 = C$, where C is a constant. [Hint: Use the Lagrangian. Ans: $x_1 = C/2, x_2 = C/8$ and the Lagrange multiplier $\lambda = C/8$.

(ii) Suppose C is set at 16. What will be the effect on the max f if there is a small change in C? [Hint: Envelop Theorem]

2. The production function of a firm is given by

$$y = x_1^{\alpha}x_2^{1-\alpha}$$

where y is output, x_1, x_2 are inputs and α $(0 < \alpha < 1)$ is a constant. Let input prices be fixed at $w_1 = 1$ and $w_2 = w > 0$. Suppose the firm's objective is to choose inputs in order to produce at least $\overline{y}$ units of output at the minimum cost.

(a) Set up the firm's problem with its constraints

(b) Solve for the units of x_1 and x_2 that solves the firm's problem.

(c) How will the cost of production change if w increases by a small amount? [Hint: Envelop Theorem]

3. Use the Kuhn–Tucker method to find the maximum value of $f(x, y) = x^2 + x + 4y^2$ subject to $2x + 2y \leq 1$, $x \geq 0$ and $y \geq 0$.

4. (i) Compute the indefinite integral $\int x^2 \ln x dx$ [Hint: Use integration by parts]

(ii) Find the integral: $\int (2/3)^x dx$.

5. (i) Compute the indefinite integral $\int (1 + x)(x - 5)^{10} dx$. [Hint: Use $u = x - 5$, then $u + 6 = x + 1$.] Ans: $I = \frac{(x-5)^{12}}{12} + \frac{6(x-5)^{11}}{11} + c$, where c is a constant of integration.

(ii) Compute the indefinite integral $I = \int x^2 e^x dx$ [Hint: You may have to integrate by parts twice]

6. Use the substitution $u = 1/x$ to show that

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = 0$$

Hint: You should first show that

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = -\int_0^\infty \frac{\ln u}{1+u^2} du$$

and use the fact that u is just a dummy variable

7. An individual with a utility function defined over per-period consumption $U(c_t)$ wishes to maximize her intertemporal utility given by

$$U_o = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to the constraint

$$c_t + s_t = y_t + (1+r)s_{t-1}$$

where $0 < \beta < 1$ is the discount factor, r is the fixed rate of interest on savings, s_t is savings, and y_t is income at time t. [Ignore the boundary conditions for now]

(i) Suppose $U(c_t) = \frac{c_t^{1-\delta}}{1-\delta}$, where $\delta \in (0,1)$ is a constant. Show that the optimal consumption decision is given by

$$c_{t+1} = [\beta(1+r)^{1/\delta}]c_t$$

(ii) Suppose $U(c_t) = \ln c_t$. Show that the optimal consumption decision is given by

$$c_{t+1} = [\beta(1+r)]c_t$$

8. Consider an individual who derives utility from the consumption of positive amounts goods x_1 and x_2 given by $U(x_1, x_2) = x_1 + x_1 x_2$. The individual has a total income of Y and the prices of the goods are given by p_1 and p_2.

(i) First, set up the PRIMAL problem (maximize utility subject to budget constraint) and solve for the Marshallian demands, x_1 and x_2. Find the indirect utility $V = V(p_1, p_2, Y)$.

(ii) Now, set up the DUAL problem (minimize expenditure subject to the constraint that utility is at least $\overline{U}$) and solve for the Hicksian demands, h_1 and h_2. Find the expenditure function $E = E(p_1, p_2, \overline{U})$.

(iii) Verify Roy's Identity from your answer in (i). Verify Shephard's Lemma from your answer in (ii).

(iv) Suppose p_1 goes up. What will be the effect on x_2? Use the Slutksy Equation to separate this into income and substitution effects.

Solutions

(1) Write the Lagrangian as:

$$L(x_1, x_2, \mu)x_1x_2 - \mu(x_1 + 4x_2 - C)$$

where μ is the Lagrange multiplier. The first order conditions are:

$$\begin{aligned} x_2 - \mu &= 0 \\ x_1 - 4\mu &= 0 \\ \text{and} \quad x_1 + 4x_2 &= C \end{aligned}$$

Then sub $x_1 = 4\mu$ and $x_2 = \mu$ into the 3rd constraint to get $8\mu = C$. The answer is: $x_1 = C/2, x_2 = C/8, \mu = C/8$.

(ii) Using the Envelop Theorem, $\frac{\partial f}{\partial C} = \mu = C/8$. Hence a unit change in C will change the optimal f by 2 if $C = 16$.

(2) The problem is as ffs:

$$\min_{x_1,x_2} x_1 + wx_2$$

subject to $x_1^{\alpha}x_2^{1-\alpha} \geq \overline{y}$.

(a) The Lagrangian is

$$L(x_1, x_2, \mu)x_1 + wx_2 - \mu(x_1^{\alpha}x_2^{1-\alpha} - \overline{y})$$

The first order conditions are:

$$\begin{aligned} 1 - \mu\alpha x_1^{\alpha-1}x_2^{1-\alpha} &= 0 \\ w - \mu(1-\alpha)x_1^{\alpha}x_2^{-\alpha} &= 0 \\ \mu(x_1^{\alpha}x_2^{1-\alpha} - \overline{y}) &= 0 \\ \text{and} \quad \mu &\geq 0 \end{aligned}$$

(b) Clearly, $\mu > 0$. So $x_1^{\alpha}x_2^{1-\alpha} = \overline{y}$. T solve, first multiply the first eqn by x_1 to get $x_1 = \mu\alpha x_1^{\alpha}x_2^{1-\alpha}$, and multiply the second eqn by x_2 to get $wx_2 = \mu(1-\alpha)x_1^{\alpha}x_2^{1-\alpha}$. Then sub into the third eqn and solve for μ (in terms of α and w). Then solve for x_1 and x_2.

(c) By Envelop Theorem, the effect of a unit change in w on the cost will be given by: $\frac{\partial \text{cost}}{\partial w} = x_2$.

3. Try this.

4. (i) Let $u = \ln x$ and $dv = x^2 dx$. Then $du = (1/x)dx$ and $v = \frac{x^3}{3}$. Using integration parts, we get

$$\begin{aligned}\int x^2 \ln x dx &= \frac{x^3}{3}\ln x - (1/3)\int x^2 dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 dx \\ &= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C\end{aligned}$$

where C is a constant of integration.

(ii) Let $y = (2/3)^x$. Then $\ln y = x\ln(2/3)$. Differentaiting with respect to x gives $\frac{1}{y}dy = \ln(2/3)dx$; or

$$dy = (2/3)^x \ln(2/3)dx$$

Therefore

$$\begin{aligned}\int (2/3)^x dx &= \frac{1}{\ln(2/3)}\int dy \\ &= \frac{y}{\ln(2/3)} + C = \frac{(2/3)^x}{\ln(2/3)} + C\end{aligned}$$

where C is a constant of integration.

5. Try this.

6. Let $u = 1/x$, then $du = -dx/x^2$; or $dx = -du/u^2$. Also, when $x = 0, u = \infty$ and when $x = \infty, u = 0$. So

$$\begin{aligned}\int_0^\infty \frac{\ln x}{1+x^2}dx &= \int_\infty^0 \frac{\ln(1/u)}{1+(1/u^2)}(-du/u^2) \\ &= \int_\infty^0 \frac{\ln u}{1+u^2}du \qquad \text{(since } \ln(1/u) = -\ln u\text{)} \\ &= -\int_0^\infty \frac{\ln u}{1+u^2}du \qquad \text{(by interchanging limits)}\end{aligned}$$

Since u is just a dummy variable, what we have is equivalent to:

$$\int_0^\infty \frac{\ln x}{1+x^2}dx = -\int_0^\infty \frac{\ln x}{1+x^2}dx$$

which implies that $2\int_0^\infty \frac{\ln x}{1+x^2}dx = 0$.

7. Notice that s_{t-1} is the pre–determined savings level at the beginning of period t, hence it is the state variable. Therefore, we can write the value function as

$$V(s_{t-1}) = \max_{s_t} U(c_t) + \beta V(s_t)$$

subject to $c_t = y_t + (1+r)s_{t-1} - s_t$ where $0 < \beta < 1$ is the discount factor, and r is the fixed rate of interest on savings. NOTICE THAT I have made s_t the choice variable. This is ok, since choosing s_t pretty much determines the choice of c_t.

Differentiate w..r.t c_t gives the f.o.c::

$$-U'(c_t) + \beta V'(s_t) = 0$$

We now need to find $V'(s_t)$. The trick is to differentiate the value function $V(s_{t-1})$ w.r.t s_{t-1}, then update the result to period t.

That is;

$$V'(s_{t-1}) = (1+r)U'(c_t),$$

where we have used the fact that $c_t = y_t + (1+r)s_{t-1} - s_t$. Update this for one period ahead gives

$$V'(s_t) = (1+r)U'(c_{t+1}).$$

This is known as that Benveniste–Scheikman Equation.

We can then sub this into the f.o.c to get

$$U'(c_t) = \beta(1+r)U'(c_{t+1})$$

This is equation linking intertemporal consumption.... proceed.

(i) Suppose $U(c_t) = \frac{c_t^{1-\delta}}{1-\delta}$, where $\delta \in (0,1)$ is a constant. Show that the optimal consumption decision is given by

$$c_{t+1} = [\beta(1+r)^{1/\delta}]c_t$$

(ii) Suppose $U(c_t) = \ln c_t$. Show that the optimal consumption decision is given by

$$c_{t+1} = [\beta(1+r)]c_t$$

(8) (i) The PRIMAL problem is to max $U(x_1,x_2) = x_1 + x_1x_2$ subject to $p_1x_1 + p_2x_2 \leq Y$. The first order conditions are

$$\begin{aligned} 1 + x_2 &= \lambda p_1 \\ x_1 &= \lambda p_2 \\ \lambda(p_1x_1 + p_2x_2 - Y) &= 0 \\ \text{and} \qquad \lambda &\geq 0. \end{aligned}$$

where λ is the Lagrange multiplier. Clearly, $\lambda \neq 0$. So the constraint binds; $p_1x_1 + p_2x_2 = Y$. Also $x_1 = \lambda p_2$ and $x_2 = \lambda p_1 - 1$, so sub this into the constraint to get

$$\lambda p_2 p_1 + (\lambda p_1 - 1)p_2 = Y$$

which implies that $\lambda = \frac{Y+p_2}{2p_1p_2}$.

This implies $x_1 = \frac{Y+p_2}{2p_1}$ and $x_2 = \frac{Y+p_2}{2p_2} - 1 = \frac{Y-p_2}{2p_2}$. Plugging this into U gives $V(p_1, p_2, Y) = \frac{(Y+p_2)^2}{4p_1p_2}$.

(ii) The DUAL problem is to min $E(x_1, x_2) = p_1x_1 + p_2x_2$ subject to $x_1 + x_1x_2 \geq \overline{U}$.

The first order conditions are

$$\begin{aligned} p_1 &= \lambda(1 + x_2) \\ p_2 &= \lambda x_1 \\ \lambda(x_1 + x_1x_2 - \overline{U}) &= 0 \\ \text{and} \qquad \lambda &\geq 0. \end{aligned}$$

where λ is the Lagrange multiplier. Clearly, $\lambda \neq 0$. So the constraint binds; $x_1 + x_1x_2 = \overline{U}$. Solving for λ gives $\lambda = \sqrt{\left(\frac{p_1p_2}{\overline{U}}\right)}$. The Hicksian demand functions are: $h_1 = \sqrt{\left(\frac{p_2\overline{U}}{p_1}\right)}$ and $h_2 = \sqrt{\left(\frac{p_1\overline{U}}{p_2}\right)} - 1$. The expenditure function is then given by

$$E(p_1, p_2, \overline{U}) = 2\sqrt{p_1p_2\overline{U}} - p_2.$$

(iii) Roy's Identity states that

$-\frac{\partial V/\partial p_i}{\partial V/\partial Y} = x_i$ for $i = 1, 2$.

Let's verify: $\frac{\partial V}{\partial p_1} = -\frac{(Y+p_2)^2}{4p_2p_1^2}$ and $\frac{\partial V}{\partial Y} = \frac{(Y+p_2)}{2p_1p_2}$. Hence,

$$-\frac{\partial V/\partial p_1}{\partial V/\partial Y} = \frac{\frac{(Y+p_2)^2}{4p_2p_1^2}}{\frac{(Y+p_2)}{2p_1p_2}} = \frac{(Y+p_2)}{2p_1} = x_1.$$

For good 2, we have $\frac{\partial V}{\partial p_2} = \frac{2p_2(Y+p_2)-(Y+p_2)^2}{4p_1p_2^2}$. Hence,

$$-\frac{\partial V/\partial p_2}{\partial V/\partial Y} = \frac{\frac{(Y+p_2)^2-2p_2(Y+p_2)}{4p_1p_2^2}}{\frac{(Y+p_2)}{2p_1p_2}} = \frac{Y+p_2}{2p_2} - 1 = \frac{Y-p_2}{2p_2} = x_2.$$

Therefore

(iii) Shephard's Lemma: states that $\frac{\partial E}{\partial p_i} = h_i$ for $i = 1, 2$.

Let's verify: $\frac{\partial E}{\partial p_1} = \sqrt{\left(\frac{p_2\overline{U}}{p_1}\right)} = h_1$ and $\frac{\partial E}{\partial p_2} = \sqrt{\left(\frac{p_1\overline{U}}{p_2}\right)} - 1 = h_2$.

(iv) The Slutsky equation: This states that

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - h_j \frac{\partial x_i}{\partial Y}$$

for $i \neq j$

Let's verify: The left hand side is $\frac{\partial x_1}{\partial p_2} = \frac{1}{2p_1}$; The RHS terms are

$$\frac{\partial h_1}{\partial p_2} = \frac{1}{2}\sqrt{\frac{\overline{U}}{p_1 p_2}}; \quad \text{and} \quad h_2 \frac{\partial x_1}{\partial Y} = \left(\sqrt{\left(\frac{p_1\overline{U}}{p_2}\right)} - 1\right) . \frac{1}{2p_1}$$

Clearly

$$\frac{\partial h_1}{\partial p_2} - h_2 \frac{\partial x_1}{\partial Y} = \frac{1}{2}\sqrt{\frac{\overline{U}}{p_1 p_2}} - \left(\sqrt{\left(\frac{p_1\overline{U}}{p_2}\right)} - 1\right) . \frac{1}{2p_1} = \frac{1}{2p_1},$$

so the Slutsky equation is verified. Hence, we are able to separate the change in quantity demanded of good 1 into subst effect ($\frac{\partial h_1}{\partial p_2}$) and income effect ($h_2 \frac{\partial x_1}{\partial Y}$).

For the second good, clearly x_2 is independent of p_1, hence $\frac{\partial x_2}{\partial p_1} = 0$. So it must be that the income and subsitution effect offset each other. Let's verify this.

$$\frac{\partial h_2}{\partial p_1} = \frac{1}{2}\sqrt{\frac{\overline{U}}{p_1 p_2}}; \quad \text{and} \quad h_1 \frac{\partial x_2}{\partial Y} = \frac{1}{2p_2}\sqrt{\frac{p_2\overline{U}}{p_1}} = \frac{1}{2}\sqrt{\frac{\overline{U}}{p_1 p_2}} \qquad \text{(offset each other)}$$