# Nilpotent orbits of a reductive group over a local field

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# Overview

# Reductive groups

$G$ connected, reductive group over the (arbitrary) field $K$.
Examples:

- $G = \mathrm{GL}(V)$ for a $K$-vector space $V$
- $G = \mathrm{Sp}(V)$ if $V$ has a non degenerate alternating form

# Levi factors

Let $H$ be a linear algebraic group over $K$.

- Assume that $R_u H$ is defined over $K$.
  - assumption *not* valid e.g. if $H = R_{K_1/K}\mathbf{G}_m$ for a purely inseparable extension $K_1/K$.
- A Levi factor of $H$ is a $K$-subgroup $M \subset H$ for which $H$ is the semidirect product $R_u H \cdot M$.
- If $K$ has char. $0$, then $H$ has a Levi factor (Mostow).
- if char. $K$ is positive, $H$ need not have a Levi factor.
  - $H = R_{K_1/K}\mathbf{G}_m$ has no Levi factor
  - if $W_2$ = Witt vectors over alg. closed $k$, let $H = \mathrm{SL}_2(W_2)$. $H$ is 6 dimensional, $R_u H$ is defined over $k$, and $H$ and has no Levi factor.

# Nilpotent orbits

- Let $G$ be reductive, and suppose $G$ satisfies some "standard hypotheses".
- when $G$ is semisimple, these amount to: the characteristic *very good* for $G$.
- Let $\mathcal{N} \subset \mathfrak{g}$ nilpotent variety.
- $G$-orbits in $\mathcal{N}$ are classified *geometrically* by Bala-Carter data $(L, Q)$: $L$ is the Levi subgroup of a parabolic of $G$, and $Q \subset L$ is a distinguished parabolic

# Structure of a nilpotent centralizer

- Fix $X \in \mathcal{N}(K) \subset \mathfrak{g}(K)$.

**Theorem**

*$C = C_G(X)$ has a Levi decomposition defined over $K$. Moreover, the following are independent of $p$ (under our standard hyp):*

- *the (geometric) root datum of a Levi factor of $C$,*
- *the (geometric) component group $C(K_{\mathrm{alg}})/C^0(K_{\mathrm{alg}})$*

- Method of proof: may suppose $K = K_{\mathrm{alg}}$.
  - let $\mathcal{A}$ be a DVR with residues $K$ and fractions of char 0.
  - let $\mathcal{G}_{/\mathcal{A}}$ be split reductive with root datum of $G$
  - find a nilpotent section $X_1 \in \mathfrak{g}(\mathcal{A})$ specializing to $X$ for which the $\mathcal{A}$ group scheme $C_{\mathcal{G}}(X_1)$ is *smooth* over $\mathcal{A}$.

# Related question:

- Now suppose that $K$ is the *fractions* of a DVR $\mathcal{A}$.
- Are the methods used in the proof of the preceding theorem useful for non-reductive smooth group schemes $\mathcal{P}_{/\mathcal{A}}$ for which $G = P_{/K}$
- in particular, what about $\mathcal{P}$ a parahoric group scheme?

# “Local” fields – notation etc.

- Let $\mathcal{A}$ be a discrete valuation ring with maximal ideal $\pi\mathcal{A}$; assume $\mathcal{A}$ is $\pi$-adically complete, assume $k = \mathcal{A}/\pi\mathcal{A}$ is *perfect*, and let $K = \operatorname{Frac}(\mathcal{A})$.
- Examples:
  - $K = k((\pi))$, $\mathcal{A} = k[\![\pi]\!]$.
  - $K$ a finite extension of $\mathbf{Q}_p$, $\mathcal{A}$ int. closure of $\mathbf{Z}_p$ in $K$

# Parahoric subgroups

Let $G$ reductive over the "local" field $K$.

- DeBacker: related nilpotent $G(K)$-orbits to the Bruhat-Tits building of $G$... subject to some conditions on char. of $k$.
- Bruhat and Tits: parahoric "subgroups" – certain smooth $\mathcal{A}$-group schemes $\mathcal{P}$ with $\mathcal{P}_{/K} = G$
- there is an $\mathcal{A}$-split torus $S$ in $\mathcal{P}$ for which $S_{/K}$ is a maximal $K$-split torus in $G$.

**Theorem (Bruhat-Tits, I think)**

*The special fiber $\mathcal{P}_{/k}$ has a unique Levi factor containing $S_{/k}$.*

# Nilpotent orbits via parahorics

- Fix $X \in \mathfrak{g}(K)$ nilpotent, and fix a cocharacter $\phi : \mathbf{G}_m \to G$ defined over $K$ and *associated with* $X$.
- Let $S$ a maximal $K$-split torus for which $\phi \in X_*(S)$.
- suppose the parahoric $\mathcal{P}$ contains (the $\mathcal{A}$-form of) $S$ and that $X \in \mathfrak{p} = \mathrm{Lie}(\mathcal{P}) = \mathrm{Lie}(\mathcal{P})(\mathcal{A})$, an $\mathcal{A}$-lattice in $\mathfrak{g}$.
- Assume the max'l reductive quotient of $\mathcal{P}_{/k}$ satisfies the standard hyp.

# Reformulation of (part of) DeBacker's argument

Let $C \subset \mathfrak{g}(K)$ linear subspace, and write $C_{/k}$ for the image of $C \cap \mathfrak{p}$ in $\mathfrak{p}_{/k} = \mathfrak{p} \otimes_{\mathcal{A}} k$. Assume:

(C1) $C$ is stable under the image of $\phi$.

(C2) $C$ is a complement to $[X, \mathfrak{g}(K)]$,

(C3) $C \cap \mathfrak{p}$ is a complement to $[X, \mathfrak{p}]$, and

(C4) $C_{/k} \cap \mathrm{Lie}(R_u \mathcal{P}_{/k})$ is a complement to $[X, \mathrm{Lie}(R_u \mathcal{P}_{/k})]$.

- If there is $C$ satisfying (C1)–(C4), the centralizer $C_{\mathcal{P}}(X)$ is a smooth group scheme over $\mathcal{A}$.
- If $p >> 0$ DeBacker uses $C = \mathrm{Lie}(C_G(Y))$ determined by a $\mathfrak{sl}_2$-triple $(X, H, Y)$ for which $H = d\phi(1)$.
- Existence of $C$ in general?!?

# Some results

Write $\mathfrak{p}^+$ for the pre-image of $\mathrm{Lie}(R_u\mathcal{P}_{/k})$ under the mapping $\mathfrak{p} \to \mathfrak{p}_{/k} = \mathrm{Lie}(\mathcal{P}_{/k})$. Assume $C$ satisfies (C1)–(C4).

### Proposition (adaptation of DeBacker/Waldspurger)

*The $G(K)$-orbit of $X$ is the nilpotent orbit of minimal dimension which intersects $X + \mathfrak{p}^+$.*

### Proposition

*There is a mapping*

$$H^1(k, C_{\mathcal{P}_{/k}}(X)) \to H^1(K, C_G(X)),$$

Note that the first Galois cohomology set of $C_{\mathcal{P}_{/k}}(X)$ identifies with that of its reductive quotient.

# Example: $\mathrm{Sp}_4$

- Let $G = \mathrm{Sp}_4 = \mathrm{Sp}(V)$ over $K$; assume char. $k$ not 2; fix a hyperbolic basis $e_1, e_2, f_1, f_2$ of $V$.
- Consider the subregular nilpotent element

$$X = \left(\begin{array}{cc|cc} 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \in \mathfrak{g}(K).$$

- Consider the parahoric $\mathcal{P}$ which is the stabilizer of the lattice $\mathcal{L} = \mathcal{A}e_1 + \mathcal{A}e_2 + \mathcal{A}\pi f_1 + \mathcal{A}f_2$ where $e_1, e_2, f_1, f_2$ is a hyperbolic basis.
- $\mathcal{P}_{/k}$ acts faithfully on $\mathcal{L}/\pi\mathcal{L} \oplus \pi^{-1}\mathcal{L}/\mathcal{L}$.
- Levi factor of $\mathcal{P}_{/k}$ is $\mathrm{SL}_2 \times \mathrm{SL}_2$, and $\dim R_u\mathcal{P}_{/k} = 4$.
- There is a $C \subset \mathfrak{g}(K)$ satisfying (C1)–(C4).

# $\mathrm{Sp}_4$ example continued

- non-zero $k$-rational nilpotent orbits for $\mathrm{SL}_2 \times \mathrm{SL}_2$ are in bijection with $k^\times/k^{\times 2} \times k^\times/k^{\times 2}$.
- the map on Galois cohomology corresponds to the correspondence:

$$\left(\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}\right) \mapsto \left(\begin{array}{cc|cc} 0 & 0 & \lambda\pi & 0 \\ 0 & 0 & 0 & \mu \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

for $\lambda k^{\times 2}, \mu k^{\times 2} \in k^\times/k^{\times 2}$.