Nilpotent centralizers in good characteristic

George McNinch

george.mcninch@tufts.edu

Tufts University

Introduction

- This talk concerns nilpotent elements of the Lie algebra of a reductive group in good characteristic.
- Let H be a connected reductive group defined over an arbitrary field F.
- Eventually we will "compare" versions of H over different fields. For this, let A be a discrete valuation ring with field of fractions K and residue field k.
  e.g. $A = \mathbf{Z}_{(p)}$, $K = \mathbf{Q}$ and $k = \mathbf{F}_p$ for a prime p.
- Let G be a split, reductive group scheme $/A$ with $\mathrm{Lie}(G) = \mathfrak{g}$.
- Write $G_{/K}$ for the generic fiber and $G_{/k}$ for the special fiber.

Strongly standard

- We wish to make some assumptions on the characteristics of K and k. I'll first make the definitions for groups over a field F.
- Consider reductive F-groups of the form

$$(*) \quad S \times T \quad \text{for } S \text{ semisimple } /F, \text{ and } T \text{ an } F\text{-torus}$$

  where $\mathrm{char}(F)$ is very good for S.
- Say that a reductive F-group H is *strongly standard* if:

  there exists an F-group H_1 of the form $(*)$, and an F-Levi subgroup M of H_1, such that there is a separable central isogeny between H and M.

...strongly standard

- The property “strongly standard” is evidently inherited by Levi subgroups of H.
- When H is strongly standard, all adjoint orbits of H on $\mathfrak{h}$ are separable.
- The terminology “strongly standard” is borrowed from the “standard hypotheses” of Jantzen.

...strongly standard

- Similarly, consider A-group of the form $(*)\quad S \times T$ where S is *split* semisimple $/A$, where T is a split A-torus, and where $\mathrm{char}(K)$ is very good for $S_{/K}$ and $\mathrm{char}(k)$ is very good for $S_{/k}$.
- A reductive group G over A will be called split strongly standard if there exists an A-group G_1 of the form $(*)$ and a splittable étale central isogeny between G and an A-split Levi subgroup of G_1.
- The fibers of a split strongly standard A-group are evidently strongly standard in the previous sense.
- $\mathrm{GL}(\mathcal{L})$ is split strongly standard for a finite rank free A-module $\mathcal{L}$.

Associated cocharacters

- For the moment, F is a field, H is a strongly standard reductive F-group and $\mathcal{N}$ is the nilpotent variety of H.
- If $X \in \mathcal{N}$ and $S < C_H(X)$ is a maximal torus, then $X \in \mathrm{Lie}(L)$ is a *distinguished* nilpotent, where $L = C_H(S)$.
- A cocharacter $\chi : \mathbf{G}_m \to H$ is associated to $X \in \mathcal{N}$ if:

  (AC1) $\mathrm{Ad}(\chi(t))X = t^2 X \; \forall t$; i.e. $X \in \mathfrak{h}(\chi; 2)$.

  (AC2) there is a maximal torus $S < C_H(X)$ so that the image of χ lies in $L' = (L, L)$ where $L = C_H(S)$.

Existence of associated cocharacters

- Let $X \in \mathcal{N}(F)$.
- If $\mathrm{char}(F) = 0$, use Jacobson-Morozov to find an F-homomorphism $\phi : \mathrm{SL}_{2/F} \to H$ with $d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$.

  Then the restriction of ϕ to the diagonal torus of SL_2 gives a cocharacter associated to X and defined over F.
- Whenever H is strongly standard, one has:

  Theorem 1: There exist cocharacters associated with X which are defined over F. Any two such are conjugate under $R_u(C_H(X))(F)$.

The instability parabolic

- Keep the preceding notation H, $X \in \mathcal{N}(F)$, etc.
- Let χ a cocharacter associated with X. Write $P(\chi) = P_H(\chi)$ for the parabolic subgroup whose Lie algebra is

$$\bigoplus_{i \geq 0} \mathfrak{h}(\chi; i).$$

- Then $P(\chi)$ depends only on X and not on the choice of a cocharacter associated to X.

...instability parabolic

- In the context of geometric invariant theory, a nilpotent element is an unstable vector in $(\mathrm{Ad}, \mathfrak{h})$. Kempf and Rousseau have associated a collection of optimal cocharacters to an unstable vector in a linear H-representation (ρ, V).
- All optimal cocharacters for an unstable vector v define the same parabolic subgroup Q, and any two optimal cocharacters are conjugate under $R_u(Q)$.
- When F is perfect, (ρ, V) is defined over F, and v is F-rational, one finds that Q is defined over F.

...instability parabolic

- A cocharacter associated with $X \in \mathcal{N}(F)$ is optimal in the above sense.
- Theorem 1 shows the instability parabolic P to be defined over F even when F is imperfect.
- The F-rationality of P in this case can also be deduced from work of Ramanan-Ramanathan (since all H-orbits in $(\mathrm{Ad}, \mathfrak{h})$ are separable).
- $C_H(X) \subset P$ since P is the instability parabolic of X.

Density

- As before, let $X \in \mathcal{N}$ with H strongly standard. Let χ be associated with X.
- There is a Levi subgroup L of H such that (1) the image of χ lies in L', and (2) $X \in \mathrm{Lie}(L)$ is distinguished.
- The instability parabolic of X *for the reductive group* L is $Q = P_L(\chi)$, and it is a distinguished parabolic subgroup of L. The element X lies in the dense (Richardson) Q-orbit on $\mathrm{Lie}\, R_u Q$.
- The P-orbit of X is dense in $\bigoplus_{i \geq 2} \mathfrak{h}(\chi; i)$.

The unipotent radical of the centralizer

- Let $X \in \mathcal{N}(F)$. Write $C = C_H(X)$ and $R = R_u(C)$.
- Separability of $\mathrm{Ad}(H)X \implies C$ is defined over F.
- There are F-groups B for which $R_u(B)$ is not defined over F.
- This problem does not occur for C. Let P be the instability parabolic of X. Then $R = R_u(P) \cap C$. Since $\mathrm{Lie}\, R_u(P) \cap \mathrm{Lie}(C) = \mathrm{Lie}\, R$, and since C and $R_u(P)$ are defined over F, the radical R is indeed defined over F.

...unipotent radical...

- Let χ be a cocharacter associated with X and defined over F. Then

$$R_u(P) = \{x \in P | \lim_{t \to 0} \operatorname{int}(\chi(t))x = 1\}.$$

  In particular, 1 is the only fixed point in $R_u(P)$ of the image of χ.

- The image of χ normalizes C and also R. Since $R = C \cap R_u(P)$, the only element of R which is fixed by the image of χ is 1.
- This permits an answer to a question of Kazhdan:
  Theorem 2: R is an F-split unipotent group. In particular, $H^1(F, R)$ is trivial.

Rational nilpotent orbits

Let H over F as before. Fix $X \in \mathcal{N}(F)$. Write $C = C_H(X)$, and write $\mathcal{O}$ for the H-orbit of X.

- Let χ be a cocharacter associated with X and over F. Let M be the centralizer in C of the image of χ. Then M is a Levi factor of C defined over F.
- Springer-Steinberg: $|C/C^o| = |M/M^o|$ is not divisible by $\mathrm{char}(F)$.
- Let $F = k((t))$ with k finite or algebraically closed. A theorem of Bruhat-Tits gives: the Galois cohomology set $H^1(F, N)$ is finite for a connected reductive N over F.

...rational nilpotent orbits

- Let $F = k((t))$ with k finite or algebraically closed, and let H be a strongly standard reductive group over F.
- **Theorem 3:** $H(F)$ has finitely many orbits in $\mathcal{N}(F)$.
- This result can fail in bad characteristic.
- Since R is split unipotent, only $C/R \simeq M$ matters for the decomposition of $\mathcal{O}(F)$ into $H(F)$ orbits.

A nice nilpotent over A

- Return now to the original setting: G is a split strongly standard reductive group over the dvr A.
- An A-parabolic subgroup $P < G$ is distinguished if and only if $P_{/K}$ and $P_{/k}$ are distinguished.
- Choose a standard A-Levi subgroup L, and let $Q < L$ be a standard distinguished A-parabolic subgroup of L.
- Let $\mathrm{der}(L)$ be the derived subgroup scheme of L; $\mathrm{der}(L)$ is a closed, semisimple A-subgroup of L.

...nice nilpotent...

- Write T for the "standard" maximal torus of L, and write $T' = \mathrm{der}(L) \cap T$.

  There is a cocharacter over A

  $$\chi : \mathbf{G}_{m/A} \to T'$$

  such that for simple roots α of L we have

  $$\langle \alpha, \chi \rangle = \begin{cases} 2 & \text{if } (\mathrm{Lie}\, R_u(Q))_\alpha \neq 0, \\ 0 & \text{if } (\mathrm{Lie}\, R_u(Q))_\alpha = 0. \end{cases}$$

...nice nilpotent...

- Write M for Levi factor of Q which is the centralizer in L of the image of χ.
- After an extension of the residue field, one may find a section $X \in \mathrm{Lie}(L)(\chi; 2)(A)$ such that

  $X_{/\overline{k}}$ lies in the open $M_{/\overline{k}}$-orbit on $\mathrm{Lie}(L)(\chi; 2)_{/\overline{k}}$ and $X_{/\overline{K}}$ lies in the open $M_{/\overline{K}}$-orbit on $\mathrm{Lie}(L)(\chi; 2)_{/\overline{K}}$.
- As a consequence, $\chi_{/k}$ is associated to $X_{/k}$ and $\chi_{/K}$ is associated to $X_{/K}$.

Comparing “good characteristics”

- Let notation be as before, and let $X \in \mathfrak{g}(A)$ be the nilpotent section chosen above.
- The Bala-Carter theorem says that the geometric nilpotent orbits of a reductive group H are in bijection with H-orbits of pairs (L, Q).
- The Bala-Carter label of $X_{/\overline{K}}$ for $G_{/\overline{K}}$ is “the same” as the Bala-Carter label of $X_{/\overline{k}}$ for $G_{/\overline{k}}$.

...comparing...

- Consider the action of the A-parabolic $P = P(\chi)$ on the free A-module $\mathcal{L} = \sum_{i \geq 2} \mathfrak{g}(\chi; i)$ (regarded as a scheme).
- The $P_{/\overline{k}}$ orbit of $X_{/\overline{k}}$ in $\mathcal{L}_{/\overline{k}}$ is dense, and the $P_{/\overline{K}}$ orbit of $X_{/\overline{K}}$ in $\mathcal{L}_{/\overline{K}}$ is dense.
- For $F = k, K$ one knows that the centralizer in $G_{/F}$ of $X_{/F}$ lies in $P_{/F}$. Thus one gets the following “for free”:
  Proposition 4: The dimension of the centralizer – and the dimension of a Levi factor of the centralizer – of a nilpotent element with given Bala-Carter label is the same for $G_{/\overline{k}}$ and for $G_{/\overline{K}}$.
- Thus these dimensions are “independent of good characteristic”.

...comparing...

- Let C be the centralizer $C_P(X)$ of X in P. C is a closed subgroup scheme of P.
- Recall that $\mathcal{L} = \sum_{i \geq 2} \mathfrak{g}(\chi; i)$. Then the orbit mapping $\alpha : P \to \mathcal{L}$ is smooth, because this is true on the fibers [the $P_{/F}$-orbit of $X_{/F}$ is separable and dense in $\mathcal{L}_{/F}$ for $F = K, k$].
- It follows that C is smooth over A.
- The centralizer of the image of χ is a smooth and closed subgroup scheme M of C [this is a general property of the centralizer of a group of multiplicative type].
- The group $C_{/F}$ may be identified with the centralizer of $X_{/F}$ in $G_{/F}$ for $F = k, K$.

...comparing...

- Thus $M_{/F}$ is a Levi factor of $C_{/F}$ for $F = k, K$.
- Let M^o be the "identity component" subgroup scheme of M. Then M^o is closed in M and reductive over A, and its fibers may be identified with the connected components of the Levi factors of $C_{/F}$ for $F = k, K$.
- Since the "geometric type" of the fiber of a reductive A-group is constant on $\mathrm{spec}(A)$, we get:

  Theorem 5: The geometric type of the connected component of a Levi factor of the centralizer of a nilpotent element with given Bala-Carter label is the same for $G_{/\overline{k}}$ and for $G_{/\overline{K}}$.