

# CAT(0) HNN-EXTENSIONS WITH NON-LOCALLY CONNECTED BOUNDARY

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ABSTRACT. The main result of this paper shows that when a certain type of HNN-extension acts geometrically on a CAT(0) space, the boundary of the space is not locally connected. The motivating example is an HNN-extension that arises from studying parabolic semidirect products of  $F_2$  and  $\mathbb{Z}$ . We show all such semidirect products act on the CAT(0) space constructed here. We explicitly identify a point of non-local connectivity in the boundary of this space as a guide to reading the proof of the main theorem.

In [MR], we show that if the group  $\Gamma$  is an amalgamated product acting geometrically on a CAT(0) space  $X$  and an orbit of the amalgamated subgroup is “quasi-convex” in  $X$ , then two group theory hypotheses about the amalgamation guarantee that  $\partial X$  is not locally connected. In particular, with the exception of  $\mathbb{Z}^n$  all right angled Artin groups satisfy the conditions of this result. When the amalgamated subgroup is abelian an orbit is always quasi-convex in  $X$ , so one can easily build groups acting geometrically on CAT(0) spaces with non-locally connected boundary by amalgamating known CAT(0) groups over abelian subgroups in the correct way.

The main result of this paper deals with HNN-extensions which act geometrically on CAT(0) spaces. Just as in the case of amalgamated products, the normal form theorem for HNN-extensions gives significant control on the structure of the Cayley graph with the standard HNN presentation. Since a quasi-isometric copy of the Cayley graph approximates the space the group acts on, there is geometry forced on the space. This in turn forces topological characteristics on the boundary much like in [MR].

The motivating example is presented in section 3. Our explanation of how to detect points of non-local connectivity in the boundary of this example is meant to introduce the ideas and techniques used in the proof of the main theorem. The example is interesting for several reasons. First, it is an HNN-extension satisfying the hypotheses of the main theorem. Second, it is a parabolic semidirect product

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of  $F_2$  with  $\mathbb{Z}$ . In fact all such parabolic semidirect products act geometrically on the space constructed in this section. The same example is used in [G] to give an example of a CAT(0) 2-complex illustrating a new notion of divergence of geodesics. The definition of divergence used in [G] is clearly seen to be a quasi-isometry invariant. The example is shown to have quadratic divergence in this new sense.

There are five sections in this paper. The second section contains definitions and known results used in the proof of the main theorem. Section four contains the necessary lemmas for understanding how the structure of the Cayley graph of an HNN-extension forces geometry on the space the group acts on. The final section contains the proof of the main theorem.

**Theorem.** *Suppose  $\alpha : A \rightarrow B$  is an isomorphism between finitely generated subgroups of the finitely generated group  $G$  and that  $G^*$  is the HNN-extension  $\langle G, t : t^{-1}at = \alpha(a), a \in A \rangle$ .*

*If  $G^*$  acts geometrically on the CAT(0) space  $X$  then  $\partial X$  is not locally connected when the following conditions are satisfied:*

- (1)  $A \neq G \neq B$
- (2)  $Ax_0$  is quasi-convex in  $X$  for some basepoint  $x_0 \in X$ .
- (3) There exists  $s \in G^*$  such that  $s^n \notin A$  for all  $n \neq 0$  and  $sAs^{-1} \subset A$ .

*Remark.* Condition (2) is independent of basepoint. The quasi-convexity of  $Ax_0$  implies the quasi-convexity of  $Bx_0$  and  $sAs^{-1} \subset A$  implies  $(t^{-1}st)B(t^{-1}s^{-1}t) \subset B$ , so  $A$  and  $B$  can be interchanged in this theorem.

## 2. Definitions and Examples

In this section, we give definitions and properties of CAT(0) spaces and their boundaries.

**Definition 2.1.** Let  $(X, d)$  be a complete, proper, geodesic metric space. If  $\triangle abc$  is a geodesic triangle in  $X$ , then we call  $\triangle \bar{a}\bar{b}\bar{c}$  in  $\mathbb{E}^2$  with the same side lengths, a *comparison triangle*. We say  $X$  satisfies the CAT(0) *inequality* if given any  $\triangle abc$  in  $X$  and a comparison triangle in  $\mathbb{E}^2$ , then for any two points  $p, q$  on  $\triangle abc$ , the corresponding points  $\bar{p}, \bar{q}$  on the comparison triangle satisfy

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

Intuitively, this says that triangles in  $X$  are at least as thin as they are in Euclidean space. Examples of such spaces include  $\mathbb{E}^n$ ,  $\mathbb{H}^n$ ,  $\mathbb{R}$ -trees, and the universal covers of compact Riemannian manifolds of nonpositive curvature.

*Remark 2.1.* It is known that if  $X$  satisfies the CAT(0) inequality, then  $X$  satisfies the following: (see [P])

- (1)  $d : X \times X \rightarrow \mathbb{R}$  is a convex function.

- (2) For any  $a, b \in X$ , there is a unique geodesic segment in  $X$  joining  $a$  and  $b$ .
- (3)  $X$  is contractible.

If  $G$  acts geometrically on a CAT(0) space  $X$ , then every  $g \in G$  acts as a *semisimple* isometry - this means  $|g| = \inf\{d(x, gx) : x \in X\}$  is attained for some  $x_0 \in X$ . The following notation and results can be found in [BH] or [BR]:

$$\text{Min}(g) = \{x \in X : d(x, gx) = |g|\}$$

If  $|g| = 0$ , then obviously  $\text{Min}(g)$  is the set of fixed points of  $g$ . If  $|g| > 0$  (i.e.  $g$  has infinite order), then  $g$  leaves a geodesic line (called an *axis*) set-wise fixed, acting as translation by  $|g|$  on this line. All such lines are parallel and  $\text{Min}(g)$  is exactly the union of all of these lines. Thus any  $\mathbb{Z}$  subgroup of  $G$  determines a line in this way. More generally, the following corollary of the Flat Torus Theorem [BH] shows how  $\mathbb{Z}^n$  subgroups of  $G$  determine a convex subset of  $X$  which is isometric to  $\mathbb{E}^n$ .

**Theorem 2.1** - [BH]. *If  $G$  is a group acting geometrically on a CAT(0) space  $X$ , and  $H \approx \mathbb{Z}^n$  a subgroup of  $G$ , then there exists a point  $x \in X$  such that the orbit  $H \cdot x$  is a lattice in an isometrically embedded copy of  $\mathbb{E}^n$  (i.e. an  $n$ -flat).*

**Definition 2.2.** Let  $X$  be a CAT(0) space and  $x_0$  a basepoint in  $X$ . Then we denote by  $\partial X$  the set of geodesic rays based at  $x_0$ .

Note that if two geodesic rays based at  $x_0$  stay a bounded distance apart in  $X$ , then they must be the same ray, by the convexity of the distance function.

If  $g$  is a semisimple isometry with  $|g| > 0$  as above, then we denote by  $g(\pm\infty)$  the endpoints of an axis for  $g$ .

To put a topology on  $\overline{X} = X \cup \partial X$  which induces the metric topology on  $X$ , consider the sequence of maps  $f_n : X \rightarrow X$  where  $f_n$  is the identity on  $B(x_0, n)$  and for any point  $x$  not in  $B(x_0, n)$ ,  $f_n(x)$  is the point on the geodesic from  $x_0$  to  $x$ , of distance  $n$  from  $x_0$ . Now  $\overline{X}$  is the the inverse limit of the sequence  $\{f_n\}$ . Changing basepoint leads to pro-isomorphic inverse sequences and a homeomorphism between the inverse limits. For details, see [BH]. The correct intuition for this topology is: two rays are close in  $\partial X$  if they travel close together for a long time.

For example,  $\partial\mathbb{E}^n = S^{n-1}$ . If  $T$  is a simplicial tree, each of whose vertices has valence at least three, then  $\partial(T) = C$  where  $C$  denotes a Cantor set.

**Definition 2.3.** Suppose  $X$  is a CAT(0) space and  $S \subseteq X$ . The *limit set* of  $S$ , denoted  $L(S)$ , is defined to be  $\overline{S} \cap \partial X$  where  $\overline{S}$  is the closure of  $S$  in  $X \cup \partial X$ . If  $p \in L(S)$ , then there is a sequence  $\{s_n\}$  of elements of  $S$  such that  $\lim_{n \rightarrow \infty} s_n = p$  where the limit is taken in  $X \cup \partial X$ .

**Definition 2.4.** A topological space  $X$  is *locally connected* if components of open sets in  $X$  are open.

**Definition 2.5.** A CAT(0) space  $X$  is *almost extendible* if there exists a constant  $E$  such that for any pair of points  $x, y$  in  $X$ , there is a geodesic ray  $r : [0, \infty) \rightarrow X$  such that  $r(0) = x$  and  $r$  passes within  $E$  of  $y$ . The number  $E$  is the *almost extendability constant*.

It is a theorem of P. Ontaneda [O] that if  $G$  acts geometrically on a CAT(0) space  $X$ , then  $X$  is almost extendible.

**Definition 2.6.** A subset  $A$  of a geodesic metric space  $X$  is *quasi-convex* if there is a  $Q > 0$  such that for  $a, b \in A$  any geodesic in  $X$  between  $a$  and  $b$  lies within  $Q$  of  $A$ . The constant  $Q$  is called the *quasi-convexity constant* for  $A$ .

There is a special case of our theorem that follows from the main theorem of [MR]. Considering that case here reduces the complexity of the proof we pursue.

**Theorem 2.2.** *Suppose  $B < A < G$ ,  $\alpha : A \rightarrow B$  is an isomorphism,  $G^* = \langle t, G : t^{-1}at = \alpha(a), a \in A \rangle$  acts geometrically on a CAT(0) space  $X$ ,  $Ax_0$  is quasi-convex in  $X$  and  $A \neq G$ . Then  $\partial X$  is not locally connected. (Similarly for  $B \neq G$  and  $A < B$ .)*

*Proof.* Let  $H = \langle t, A \rangle < G^*$ , then  $G^* = H *_A G$ ,  $A$  has index  $\geq 2$  in  $G$  and  $A$  has infinite index in  $H$ . So  $G^*$  satisfies the hypothesis of Theorem [MR] (with  $t^{-1} = s$ ).  $\square$

From this point on we assume  $A \not\subset B$  and  $B \not\subset A$ .

*Remark 2.2.* The following was conjectured in [MR]: Suppose  $A < G$  are groups,  $G$  acts geometrically on a CAT(0) space  $X$  and  $Ax_0$  is quasi-convex in  $X$ , then if there exists  $s \in G$  such that  $sAs^{-1} \subset A$  then in fact  $sAs^{-1} = A$ .

### 3. THE MOTIVATING EXAMPLE

The following example illustrates some of the ideas used in the proof of the main theorem. Suppose  $G$  has presentation

$$(1) \quad G = \langle x, y, z \mid [x, y] = 1, z^{-1}xz = y \rangle.$$

Then  $G$  is the HNN extension with base  $A = \mathbb{Z} \oplus \mathbb{Z}$  generated by  $x, y$ . The associated subgroups are  $C_1 = \langle x \rangle$  and  $C_2 = \langle y \rangle$  and the stable letter  $z$  conjugates  $C_1$  to  $C_2$  via the relation  $z^{-1}xz = y$ .

We came across this example while investigating semidirect products of  $F_2$ , the free group on two letters, with  $\mathbb{Z}$ . The group  $G$  described here is one of these semidirect products. In fact, every parabolic semidirect product of  $F_2$  with  $\mathbb{Z}$  can be viewed as an HNN extension of  $\mathbb{Z} \oplus \mathbb{Z}$  over  $\mathbb{Z}$  because of the lemma given below. This particular group has also been studied in [G] where it is shown to have quadratic divergence.

**Lemma 3.1.** *Let  $\lambda : F_2 \rightarrow F_2$  be given by  $a \mapsto ab$  and  $b \mapsto b$  where  $a, b$  generate  $F_2$ . Then the semidirect product of  $F_2$  with  $\mathbb{Z}$  determined by  $\lambda^n$ , denoted  $G_n$  has presentation  $\langle z, x, y : z^{-1}xz = y^n x^{-n+1}, [x, y] \rangle$  and thus is an HNN-extension with base  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $x$  and  $y$  with associated  $\mathbb{Z}$  subgroups generated by  $x$  and  $y^n x^{-n+1}$ .*

*Proof.* The semidirect product  $G_n$  has presentation

$$\langle a, b, t \mid tat^{-1} = ab^n, tbt^{-1} = b \rangle.$$

Now let  $z = a$ ,  $x = t$ ,  $y = bt$ . This implies  $b = yt^{-1} = yx^{-1}$ . We have

$$\begin{aligned} G_n &= \langle z, x, y \mid xzx^{-1} = z(yx^{-1})^n, x(yx^{-1})x^{-1} = yx^{-1} \rangle \\ &= \langle z, x, y \mid z^{-1}xz = (yx^{-1})^n x, xyx^{-1} = y \rangle \\ &= \langle z, x, y \mid z^{-1}xz = y^n x^{-n+1}, xy = yx \rangle. \end{aligned}$$

There are three types of semidirect products of  $F_2$  with  $\mathbb{Z}$  depending on the type of matrix in  $GL(2, \mathbb{Z})$  that the automorphism defining the semidirect product determines. These are classified by the trace squared of the matrix and it is known that (up to conjugation) the parabolic ones are given by the powers of  $\lambda$  from the above theorem. We use the next lemma (the proof of which will be left to the reader) to reduce the problem of studying all parabolics to studying only the one determined by  $\lambda$ .

**Lemma 3.2.** *For any group  $G$ , there is a monomorphism  $\iota_p : G \rtimes_{\phi^p} \mathbb{Z} \rightarrow G \rtimes_{\phi} \mathbb{Z}$  whose image is a normal subgroup of index  $p$ .*

It is known (see [Br] for instance) that semi-direct products of  $F_2$  and  $\mathbb{Z}$  are CAT(0) groups. We construct a CAT(0) space for  $G$  using the HNN presentation for  $G$  and the gluing theorem below from [BH]. Then since any parabolic is a finite index subgroup of the one here, all parabolics act geometrically on this space.

**Theorem 3.3.** *Let  $X$  and  $A$  be metric spaces of curvature  $\leq \kappa$  where  $\kappa \leq 0$ . If  $A$  is compact and  $\phi, \psi : A \rightarrow X$  are local isometries, then the quotient of  $X \cup (A \times [0, 1])$  by the equivalence relation generated by  $(a, 0) \equiv \phi(a)$  and  $(a, 1) \equiv \psi(a) \forall a \in A$  also has curvature  $\leq \kappa$ .*

The phrase “curvature  $\leq \kappa$ ” simply means that the spaces satisfy the CAT(0) inequality locally. We apply the theorem with  $X$  a torus,  $A$  a circle, and the isometries  $\phi$  and  $\psi$  from  $A$  to the circles which represent the standard generators  $x$  and  $y$  in the fundamental group of the torus. Then  $G$  acts geometrically on the (CAT(0)) universal cover of the resulting quotient space. We change notation here and denote this universal covering space by  $X$ . Observe that  $X$  is the universal cover of the “standard” 2-complex corresponding to the presentation (1). Hence

the 1-skeleton is the Cayley graph of (1). In particular, this defines directed labels on the edges of  $X$ . Each label is an  $x$ ,  $y$  or  $z$ .

$X$  can be viewed as a collection of planes (copies of the universal cover of the original torus) glued together along lines with strips between them via the identifications given by the HNN description of the group  $G$ . Start with a plane  $P_0$  whose origin will be the basepoint  $*$  of  $X$  so that  $P_0$  contains the orbit of the  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $x$  and  $y$ . The edge labeling of this plane is the standard  $xy$  grid (we can assume that  $x$  and  $y$  act as translations of length 1 in perpendicular directions inside  $P_0$ ). There is a vertical strip of height one along the  $x$  axis of  $P_0$  with each of the vertical edges labeled by a  $z$ . Because of the identifications in  $G$ , the top edge of this strip is labeled by  $y$  edges. These  $y$  edges lie in a plane, denoted  $P_1$  which is the image of  $P_0$  under the action of  $z$ . Thus we have two planes, one glued to the top and the other to the bottom of a strip. Notice that the  $x$ -axis in  $P_0$  is parallel to the  $y$ -axis in  $P_1$ .

In  $P_0$ , denote the  $x$ -axis by  $A_x$  and the  $y$ -axis by  $A_y$ . Since  $y$  commutes with  $x$ ,  $yA_x$  is another axis for  $x$  which is disjoint from and parallel to  $A_x$ . Thus we have a family of parallel lines in  $P_0$  given by  $y^n A_x$ . There is a strip and plane glued to each of these lines as above. Consider the strip with base  $y^n A_x$ . The plane at the top of this strip is  $y^n P_1$ . The planes  $P_1$  and  $y^n P_1$  are disjoint, but “share a common direction”. I.e. the line  $zA_y$  in  $P_1$  is parallel to the line  $y^n zA_y$  in  $y^n P_1$ .

See Figure 1 for details.

Iterate this process everywhere via the action of  $G$  - i.e. in each plane there is an  $x$ -axis (actually the image of  $A_x$  under some group element) so we glue another strip with a plane on top where the top of the strip has  $y$  edge labels in the new plane and the vertical edges in the strip are labeled by  $z$ .

This is not all of  $X$  because we have not considered the action of negative powers of  $z$  on the planes we’ve constructed, however this is sufficient to see the non-local connectivity of certain points in  $\partial X$ . There is another “half” of  $X$  which is constructed via the relation  $zyz^{-1} = x$  which lies below the part we have described above.

Each plane we have described contributes a circle to  $\partial X$ . Denote the boundary circle for  $P_n \equiv z^n P_0$ , by  $S_n$ . Because  $P_0$  and  $P_1$  share a common direction,  $S_0$  and  $S_1$  have two points in common - likewise for  $S_n$  and  $S_{n+1}$ . In  $P_0$ , the endpoints of  $A_x$ , denoted  $x^{\pm\infty}$ , are the same as the endpoints for  $zA_y$  (the  $y$ -axis in  $P_1$ ). and so  $x^{\pm\infty} = zy^{\pm\infty}$ . These are the only two points in  $S_0 \cap S_1$ . The points  $zx^{\pm\infty}$  are also in  $S_1$ , but these are the same as  $z^2 y^{\pm\infty}$  in  $S_2$ . Iterate this process under the action of  $z$  on  $\partial X$ , to get a collection of circles which limits to a single point, namely  $z^\infty$ . Precisely,

$$\lim_{n \rightarrow \infty} z^n S_0 = \lim_{n \rightarrow \infty} S_n = z^\infty.$$

There is a similar collection which limits to  $z^{-\infty}$  obtained from the “other half” of  $X$  mentioned above. See Figure 2 for how this works.

Next we observe a “folding” of the planes  $y_n z P_0$  (pictured in Figure 3). By this we mean the  $y$ -axis of  $y^n z P_0$  is parallel to  $A_x$  for all  $n$  (so the  $y$ -axis of  $y^n z P_0$  limits to  $x^{\pm\infty}$  in  $y^n z S_0$ ) and for “large”  $n$  the two boundary points of the  $x$ -axis of  $y^n z P_0$  (namely  $y^n z x^{\pm\infty}$ ) are “close” in  $y^n z S_0$ . In fact,

$$\lim_{n \rightarrow \infty} y^n z x^\infty = \lim_{n \rightarrow \infty} y^n z x^{-\infty} = y^\infty.$$

To see this last equality, consider the geodesic edge paths  $r_n^+$  and  $r_n^-$  beginning at the base point  $* \in P_0$  with edge labels  $\langle y^n, z, x, x, \dots \rangle$  and  $\langle y^n, z, -x, -x, \dots \rangle$  respectively. The limit point of  $r_n^+$  (respectively  $r_n^-$ ) is  $y^n z x^\infty$  (respectively  $y^n z x^{-\infty}$ ). Now simply observe that the geodesic edge paths  $r_n^+, r_n^-$  and  $\langle y, y, \dots \rangle$  at  $*$  coincide for  $n$ -edges before diverging.

There is a similar “folding” for the negative powers of  $y$  acting on  $z S_0 \equiv S_1$ .

Now it is easy to see that  $y^\infty$  is a point of non-local connectivity. Indeed, consider the sequence  $\{y^n z^\infty\}$  in  $\partial X$ . Clearly,  $\lim_{n \rightarrow \infty} y^n z^\infty = y^\infty$  since the path labeled by  $y^n$  followed by the ray labeled with  $z$  edges is a geodesic to  $y^n z^\infty$  which follows the geodesic to  $y^\infty$  for  $n$  steps. Also, for  $m \neq n$ , the points  $y^n z^\infty$  and  $y^m z^\infty$  are in different path components of  $\partial X - \{x^{\pm\infty}\}$ .

In this example,  $y$  is playing the role of  $s$ ,  $\langle x \rangle$  is the subgroup  $A$ , and  $\langle y \rangle$  is the subgroup  $B$  in the statement of the main theorem.

#### 4. THE NECESSARY LEMMAS

Suppose  $A$  is generated by the finite set  $S_A$ ,  $B$  generated by the finite set  $S_B$  and  $G$  generated by the finite set  $S_G$ , where  $S_A \cup S_B \subset S_G$ . Let  $\Gamma = \Gamma(G^*, S)$  denote the Cayley graph of  $G^*$  with respect to the finite generating set  $S = S_G \cup \{t\}$ . Each edge of  $\Gamma$  is directed and labeled by an element of  $S$ . If  $a \in A$  then there is an edge path  $\alpha$  from  $1 \in \Gamma$  to  $a \in \Gamma$ , each of whose edges is labeled by an element of  $S_A^{\pm 1}$ . Furthermore, if  $\bar{a}$  is the element of the free group  $F(A)$ , defined by  $\alpha$ , then the natural map of  $F(A)$  to  $A$  takes  $\bar{a}$  to  $a$ . Similarly for  $b \in B$  and  $g \in G$ . We make extensive use of edge paths in  $\Gamma$  labeled by  $S^{\pm 1}$ .

Since all of our groups are CAT(0) groups, they are finitely presented (see [BH]).

Let  $x_0 \in \text{Min}(s)$ , for  $s$  as in the main theorem and let  $x_i \equiv s^i x_0$ . Let  $q : \Gamma \rightarrow X$  be a continuous map such that  $q(g) = g x_0$  for all  $g \in G$  and  $q$  commutes with the action of  $G$  on  $\Gamma$  and  $X$ . The map  $q$  is a quasi-isometry. Say that  $\frac{1}{\lambda} d_\Gamma(x, y) - \epsilon \leq d_X(q(x), q(y)) \leq \lambda d_\Gamma(x, y) + \epsilon$  for all  $x, y \in \Gamma$ . Assume the metric on  $X$  is normalized so that the diameter of  $q(e)$  is  $\leq 1$  for any edge  $e$  of  $\Gamma$ .

If  $A x_0$  is quasi-convex in  $X$  and  $g \in G$  then  $g A x_0$  is quasi-convex in  $X$  (with the same quasi-convexity constant). Also,  $g$  is a vertex of  $\Gamma$ , and  $q(g) = g x_0 \in g A x_0$ . When  $g = s^n$  we have  $s^n A x_0 \subset A s^n x_0 = A x_n$ . If  $\mu$  is a geodesic in  $X$  from  $x_0$  to  $x_n$  then for any  $a \in A$ ,  $a \mu$  is a path of diameter  $d_X(x_0, x_n)$  between  $a x_0$  and  $a x_n$ . Hence  $L(A x_0) = L(A x_n)$  for all  $n$ .

*Terminology.* Suppose  $\alpha \equiv \langle e_1, \dots, e_n \rangle$  is an edge path in  $\Gamma$  from  $v$  to  $w$ . Each  $e_i$  is labeled by an element of  $S^{\pm 1}$ . If  $t_i \in S^{\pm 1}$  labels  $e_i$ , then  $v$  and  $\langle t_1, \dots, t_n \rangle$  uniquely identify  $\alpha$  and we say  $\alpha = \langle t_1, \dots, t_n \rangle$  at  $v$ . Note that in  $G^*$ ,  $w = vt_1 \dots t_n$ . If  $g \in G^*$  then  $g(\langle t_1, \dots, t_n \rangle$  at  $v$ ) =  $\langle t_1, \dots, t_n \rangle$  at  $gv$ , an edge path in  $\Gamma$  from  $gv$  to  $gvt_1 \dots t_n$ .

**Definition 4.1.** Let  $u \equiv \langle u_1, u_2, \dots \rangle$  be an edge path (finite or infinite) in  $\Gamma$ . Consider the *standard* decomposition of  $u$  as  $\langle u_1, u_2, \dots \rangle$  where  $u_i$  is a maximal subpath of  $u$  of one of the following three types:

- (1) A path with a single edge labeled  $t$ .
- (2) A path with a single edge labeled  $t^{-1}$ .
- (3) A path in which every edge is labeled by an element of  $S_G^{\pm 1}$ .

For  $u$  to be *reduced*, we require  $u$  to have no subpaths of the form  $\langle t, t^{-1} \rangle$ ,  $\langle t^{-1}, t \rangle$ ,  $\langle t^{-1}, a, t \rangle$  where  $a$  is a path representing an element of  $A$  or  $\langle t, b, t^{-1} \rangle$  where  $b$  is a path representing an element of  $B$ .

**Lemma 4.1.** *If there exists  $s \in G^*$  such that  $sAs^{-1} \subset A$  and  $s^n \notin A$  for all  $n > 0$  then one of the following holds:*

i) *There exists  $g \in G$  such that  $g^k \notin A$  for  $k \neq 0$  and  $gAg^{-1} \subset A$ . (In this case we replace  $s$  by  $g$ .)*

ii) *There exists  $g \in G$  such that  $g^k \notin B$  for  $k \neq 0$  and  $gBg^{-1} \subset B$ . Now  $tg^kt^{-1} \notin A$  for  $k \neq 0$  and  $tgt^{-1}Atg^{-1}t^{-1} \subset A$ . (In this case we replace  $s$  by  $tgt^{-1}$ .)*

iii) *There exists  $g \in G$  such that  $gAg^{-1} \subset B$  and so  $tgAg^{-1}t^{-1} \subset A$ . Observe that  $g \notin A \cup B$  as  $A \not\subset B$ . (In this case we replace  $s$  by  $tg$ .)*

iv) *There exists  $g \in G$  such that  $gBg^{-1} \subset A$  and so  $gt^{-1}Atg^{-1} \subset A$ . Observe that  $g \notin A \cup B$  as  $B \not\subset A$ . (In this case we replace  $s$  by  $gt^{-1}$ .)*

v) *There exists  $g, h \in G$  such that  $g \notin A$  but  $g^k \in A$  for some  $k > 0$ ,  $h \notin B$  but  $h^m \in B$  for some  $m > 0$ ,  $gAg^{-1} = A$ ,  $hBh^{-1} = B$ , and so  $tht^{-1}gAg^{-1}th^{-1}t = A$ . (In this case we replace  $s$  by  $tht^{-1}g$ .)*

*Note that in case v), we need  $g \notin A$  and  $h \notin B$  to be certain that  $(tht^{-1}g)^n \notin A$  for all  $n \neq 0$ .*

*Proof.* Choose  $s$  such that  $sAs^{-1} \subset A$  and  $s^k \notin A$  for all  $k \neq 0$ . Suppose  $s = g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n$  is reduced. For any  $a \in A$ ,  $g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_nag_n^{-1}t^{-\epsilon_n} \dots t^{-\epsilon_1}g_0^{-1} \in A$ . Hence either:

- 1)  $\epsilon_n = 1$  and  $g_nag_n^{-1} \in B$  for all  $a \in A$  or
- 2)  $\epsilon_n = -1$  and  $g_nag_n^{-1} \in A$  for all  $a \in A$ .

If 1) occurs then we are in case iii).

If 2) occurs and  $g_n^k \notin A$  for  $k \neq 0$ , then we are in case i).

Hence we need only consider when 2) occurs and  $g_n^k \in A$  for some  $k \neq 0$ . In this case:

$$g_nAg_n^{-1} = A \text{ and so } t^{-1}g_nAg_n^{-1}t = B$$

If  $n = 1$ ,  $s = g_0t^{-1}g_1$  and  $t^{-1}g_1Ag_1^{-1}t = B$  so  $g_0Bg_0^{-1} = A$  and we are in Case iv). We assume  $n > 1$ .

Either:

3)  $\epsilon_{n-1} = -1$  and  $g_{n-1}bg_{n-1}^{-1} \in A$  for all  $b \in B$  or

4)  $\epsilon_{n-1} = 1$  and  $g_{n-1}bg_{n-1}^{-1} \in B$  for all  $b \in B$ .

If 3) occurs then we are in case iv).

If 4) occurs and  $g_{n-1}^m \notin B$  for  $m \neq 0$  then we are in case ii).

So we need only consider when both 2) and 4) occur,  $g_n^k \in A$ ,  $g_{n-1}^m \in B$  for some  $k \neq 0$  and  $m \neq 0$ ,  $g_nAg_n^{-1} = A$ , and  $g_{n-1}Bg_{n-1}^{-1} = B$ .

Now  $g_{n-1}t^{-1}g_nAg_n^{-1}tg_{n-1}^{-1} = B$ . As  $s$  is reduced and  $n > 1$ ,  $g_{n-1} \notin B$ . If  $g_n \notin A$  then we are in case v).

We now consider the possibility that  $g_n \in A$ . (In this case  $tg_{n-1}t^{-1}g_nAg_n^{-1}tg_{n-1}^{-1}t^{-1} = A$ .)

When  $n = 2$ ,  $s = g_0tg_1t^{-1}g_2$  and  $g_0Ag_0^{-1} \subset A$ . If  $g_0^k \notin A$  for  $k \neq 0$ , then we are in Case i). If  $g_0^k \in A$  for some  $k \neq 0$  then  $g_0Ag_0^{-1} = A$  and so  $sAs^{-1} = A$ . If  $g_0 \in A$  then  $s = tgt^{-1}$  for some  $g \in G$  and  $tgBg^{-1}t^{-1} = A$ . Then  $gBg^{-1} = B$ . As  $s^k \notin A$  for  $k \neq 0$ ,  $g^k \notin B$  for  $k \neq 0$  and we are in Case ii). We need only consider  $g_0 \notin A$ . Now  $tg_1t^{-1}g_0Ag_0^{-1}tg_1^{-1}t^{-1} = A$ , so we are in Case v).

We need only consider when:

$$n > 2, g_n \in A, g_{n-1} \notin B, \text{ but } g_{n-1}^m \in B \text{ for some } m > 1, \\ g_{n-1}Bg_{n-1}^{-1} = B \text{ and } tg_{n-1}t^{-1}Atg_{n-1}^{-1}t^{-1} = B$$

Either:

5)  $\epsilon_{n-2} = 1$  and  $g_{n-2}ag_{n-2}^{-1} \in B$  for all  $a \in A$  or

6)  $\epsilon_{n-2} = -1$  and  $g_{n-2}ag_{n-2}^{-1} \in A$  for all  $a \in A$ .

If 5) holds then we are in case iii).

If 6) holds and  $g_{n-2}^k \notin A$  for  $k \neq 0$  then we are in case i).

If 6) holds and  $g_{n-2}^k \in A$  for some  $k \neq 0$  then  $g_{n-2}Ag_{n-2}^{-1} = A$ .

Now  $g_{n-2} \notin A$  as  $g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n$  is reduced and  $n > 2$  and  $tg_{n-1}t^{-1}g_{n-2}Ag_{n-2}^{-1}tg_{n-1}^{-1}t^{-1} = A$  so we are in Case v).  $\square$

**Lemma 4.2.** *Suppose  $u$  is a reduced edge path in  $\Gamma$  with standard decomposition  $\langle u_1, u_2, \dots, u_n \rangle$ . Where,  $u_1$  and  $u_n$  do not represent elements of  $A$ . Set  $v_{k-1}$  equal to the initial point of  $u_k$  and  $v_n$  equal to the end point of  $u$ . Then the edge path from  $v_0A$  to  $v_nA$  must have at least as many  $t$  and  $t^{-1}$  occurrences as  $u$  does and for  $1 \leq k \leq n$  must intersect*

- (1)  $v_{k-1}A$  if  $u_k = t$  and
- (2)  $v_kA$  if  $u_k = t^{-1}$ .

*Proof.* Without loss, assume we begin with a reduced edge path between  $v_0a_1$  and  $v_na_2$  where  $a_i \in A$ . (Otherwise pieces of the path can be replaced by edge paths labeled by letters in  $S_A^{\pm 1}$  or by letters in  $S_B^{\pm 1}$ . [LS].)

The result now follows for reduced edge paths by Theorem IV.2.1 of [LS].  $\square$

**Lemma 4.3.** *Under the hypotheses of lemma 4.2, if there are at least  $K$  occurrences of  $t$  and  $t^{-1}$  in  $\langle u_1, u_2, \dots, u_n \rangle$  then*

$$d_X(v_0Ax_0, v_nAx_0) \geq \frac{1}{\lambda}K - \epsilon.$$

*Proof.* By lemma 4.2,  $d_\Gamma(v_0A, v_nA) \geq K$ . Now use the fact that  $q$  is a  $(\lambda, \epsilon)$ -quasi-isometry.  $\square$

By lemma 4.1 there are five possible reduced forms for  $s$ :

*Form 1.*  $g$  where  $g^k \notin A$  for  $k \neq 0$ .

*Form 2.*  $tgt^{-1}$  where  $tg^kt^{-1} \notin A$  for  $k \neq 0$ .

*Form 3.*  $tg$  where  $g \notin A \cup B$ .

*Form 4.*  $gt^{-1}$  where  $g \notin A \cup B$ .

*Form 5.*  $tht^{-1}g$  where  $g \notin A$  and  $h \notin B$ .

Now we construct an edge path ray  $r_0$  in  $\Gamma$  with initial point 1 so that if we let  $r_i$  be the translate of  $r_0$  by  $s^i$  we will have that the rays  $r_i, r_j$  and the segment  $\langle s, s, \dots, s \rangle$  from  $s^i$  to  $s^j$  define a bi-infinite reduced edge path.

If  $s$  has Form 1 then let  $r_0$  be  $\langle t, t, \dots \rangle$ .

If  $s$  has Form 2, 3, 4 or 5 then let  $r_0$  be  $\langle t^{-1}, t^{-1}, \dots \rangle$ .

See Figure 4 for the possibilities.

For  $i < j$  the reduced bi-infinite edge path referred to above is labeled by:

1.  $\langle \dots, t^{-1}, t^{-1}, g^{j-1}, t, t, \dots \rangle$  if  $s$  has Form 1.
2.  $\langle \dots, t, t, g^{j-1}, t^{-1}, t^{-1}, \dots \rangle$  if  $s$  has Form 2.
3.  $\langle \dots, t, t, g, t, g, \dots, t, g, t^{-1}, t^{-1}, \dots \rangle$  if  $s$  has Form 3.
4.  $\langle \dots, t, t, g, t^{-1}, g, t^{-1}, \dots, g, t^{-1}, t^{-1}, \dots \rangle$  if  $s$  has Form 4.
5.  $\langle \dots, t, t, h, t^{-1}, g, t, h, t^{-1}, g, \dots, t, h, t^{-1}, g, t^{-1}, t^{-1}, \dots \rangle$  if  $s$  has Form 5.

Now lemma 4.2 implies:

**Lemma 4.4.** *Any edge path between  $r_i(n_1)$  and  $r_j(n_2)$  ( $i < j$ ) must pass thru each of  $r_i(n)A$  for all  $1 \leq n \leq n_1$  and  $r_j(n)A$  for all  $1 \leq n \leq n_2$  and*

1.  $s^iA$  and  $s^jA$ , if  $s$  has form 1.
2.  $s^iA$  and  $s^jA$ , if  $s$  has form 2.
3.  $s^iA, \dots, s^{j-1}A$  if  $s$  has form 3.
4.  $s^{i+1}A, \dots, s^jA$  if  $s$  has form 4.
5.  $s^iA, \dots, s^{j-1}A$  if  $s$  has form 5.  $\square$

See Figure 5 for Cases 1 and 3. Case 2 is essentially the same as Case 1 and Cases 4 and 5 are essentially the same as Case 3.

*Notation.* Suppose  $Y$  is a metric space and  $A \subset Y$ . Then for  $K \geq 0$ ,  $N_K(A) \equiv \{y \in Y : d(y, A) \leq K\}$ . The distance function used will be evident.

**Lemma 4.5.** *Suppose  $G^* = \langle G, t : t^{-1}At = B \rangle$  acts geometrically on a CAT(0) space  $X$ . There exists an  $M > 1$  such that for every  $i < j$ , any path from  $qr_i(n_1)$  to  $qr_j(n_2)$  must intersect each of  $N_M(r_i(n)Ax_0)$  for  $1 \leq n \leq n_1$ ,  $N_M(r_j(n)Ax_0)$  for  $1 \leq n \leq n_2$  and*

Each of  $N_M(s^i Ax_0)$  and  $N_M(s^j Ax_0)$  or  
 Each of  $N_M(s^{i+1} Ax_0), \dots, N_M(s^j Ax_0)$  or  
 Each of  $N_M(s^i Ax_0) \dots N_M(s^{j-1} Ax_0)$ .

*Proof.* This proof is analogous to the proof of Lemma 3.4 in [MR]. Since the action of  $G^*$  on  $X$  is cocompact, there exists an  $N > 0$  such that  $B(N, G^*x_0) = X$ .

As  $\frac{1}{\lambda}d_\Gamma(g, h) - \epsilon \leq d_X(gx_0, hx_0)$  for all  $g, h \in G^*$ , if  $d_X(gx_0, hx_0) \leq 2N + 1$ , then there is an edge path in  $\Gamma$  between  $g$  and  $h$  with length less than  $\lambda(2N + 1 + \epsilon)$ .

Define  $M = N + \lambda(2N + 1 + \epsilon)$ .

The idea is to approximate a given path from  $qr_i(n_1)$  to  $qr_j(n_2)$  by the image under  $q$  of an edge path in  $\Gamma$  and then use lemma 4.4.

Let  $p : [0, 1] \rightarrow X$  be a path between  $gx_0$  and  $hx_0$ , where  $g = r_i(n_1)$  and  $h = r_j(n_2)$ . By the uniform continuity of  $p$ , there is a partition  $0 = k_0 < k_1 < \dots < k_m = 1$  of  $[0, 1]$  such that for each  $n \in \{0, 1, \dots, m-1\}$  the diameter of  $p([k_n, k_{n+1}])$  is  $\leq 1$ . For each  $n \in \{1, \dots, m-1\}$ , choose  $g_n \in G^*$  such that  $d_X(p(k_n), g_n x_0) \leq N$ . (Let  $g_0 = g$  and  $g_m = h$ .) For any  $n$  we have:

$$\begin{aligned}
 d_X(g_n x_0, g_{n+1} x_0) &\leq d_X(g_n x_0, p(k_n)) + d_X(p(k_n), p(k_{n+1})) \\
 &\quad + d_X(p(k_{n+1}), g_{n+1} x_0) \leq 2N + 1
 \end{aligned}$$

We know we can join  $g_{n-1}$  and  $g_n$  by an edge path  $\alpha_n$ , in  $\Gamma$ , of length  $\leq \lambda(2N + 1 + \epsilon)$ . Since we have normalized,  $q(\alpha_n)$  is a path between  $g_{n-1}x_0$  and  $g_n x_0$  of diameter  $\leq \lambda(2N + 1 + \epsilon)$ . Now the  $\Gamma$  edge path  $\langle \alpha_1, \dots, \alpha_m \rangle$  begins at  $g \equiv g_0$  on  $r_i$  and ends at  $h \equiv g_m$  on  $r_j$ . Apply lemma 4.4 to obtain an  $a \in A$  such that  $s^n a$  and  $r_j(k)ax_0$  are on  $\langle \alpha_1, \dots, \alpha_m \rangle$  for the various possibilities in each case. Thus  $s^n ax_0$  and  $r_j(n)ax_0$  lie on  $q(\alpha_u)$  for some  $u$ . We have:

$$\begin{aligned}
 d_X(s^n ax_0, p(k_u)) &\leq d_X(s^n ax_0, g_u x_0) + d_X(g_u x_0, p(k_u)) \\
 &\leq \text{diam}_X(q\alpha_u) + N \leq \lambda(2N + 1 + \epsilon) + N
 \end{aligned}$$

as needed. Similarly for  $r_j(n)ax_0$   $\square$

Now as  $r_i$  defines an infinite reduced word, it eventually must be arbitrarily far from  $s^j A$  for any  $j$ . Hence lemma 4.5 implies that for large  $K$ ,  $qr_i([K, \infty))$  and  $qr_j([K, \infty))$  are in different path components of  $X - N_M(s^i Ax_0)$  or  $X - N_M(s^j Ax_0)$

As  $s^n \notin A$  for all  $n \neq 0$ , the high powers of  $s$  get arbitrarily far from  $A$  in  $\Gamma$ . As  $\Gamma$  and  $X$  are quasi-isometric, for large  $p$ ,  $x_p \equiv s^p x_0$  is far from  $Ax_0$

The quasi-convexity of  $Ax_0$  is used in the next two lemmas.

**Lemma 4.6.** *Keeping the same hypotheses and notation as above, suppose  $Ax_0$  is quasi-convex in  $X$  with constant  $Q$ . Then the limit set of  $C$ , i.e.  $L(Ax_0)$ , in  $\partial X$  does not contain  $s(\pm\infty)$ .*

*Proof.* Suppose  $s(+\infty) \in L(Ax_0)$ . Let  $\{a_n\}$  be a sequence of elements of  $A$  such that  $\lim_{n \rightarrow \infty} a_n x_0 = s(+\infty)$ . Then the segments  $[x_0, a_n x_0]$  must converge to the

ray  $[x_0, s(+\infty)]$ . For each  $n$ ,  $[x_0, a_n x_0] \subseteq N_Q(Ax_0)$  by quasi-convexity. Thus the ray  $[x_0, s(+\infty)]$  is also contained in  $N_Q(Ax_0)$ . Since  $x_0 \in \text{Min}(s)$ , this ray is just the positive half of an axis for  $s$ , thus for all  $p \in \mathbb{N}$ ,  $s^p x_0$  lies on this ray and thus within  $Q$  of  $Ax_0$ , a contradiction.  $\square$

In all that follows,  $E$  is our almost extendability constant,  $Q$  is the quasi-convexity constant for  $Ax_0$  in  $X$ ,  $M$  is defined in lemma 4.5 and  $\lambda$  is the quasi-isometry constant of  $q$ .

**Lemma 4.7.** *Suppose  $H > 0$  and  $g \in G$ . If  $r$  is a geodesic ray with  $r(0) \in N_H(gAx_0)$  and  $t \in [0, \infty)$  is such that  $r(t) \notin N_{H+Q}(gAx_0)$  then  $r([t, \infty))$  does not intersect  $N_H(gAx_0)$ .*

*Proof.* Suppose  $r(u) \in N_H(gAx_0)$  for some  $u > t$ . Choose  $a, a' \in A$  such that  $d_X(r(0), gax_0) \leq H$  and  $d_X(r(u), ga'x_0) \leq H$ . Consider the geodesic rectangle with vertices  $gax_0, r(0), r(u), ga'x_0$ . Since the two sides  $[gax_0, r(0)]$  and  $[r(u), ga'x_0]$  have length  $\leq H$ , the side  $[r(0), r(u)]$  must be in  $N_H([gax_0, ga'x_0])$  by convexity of the distance function. By quasi-convexity of  $gAx_0$ ,  $[gax_0, ga'x_0]$  is contained in  $N_Q(gAx_0)$  as well, thus  $[r(0), r(u)] \subseteq N_{Q+H}(gAx_0)$  a contradiction.  $\square$

## 5. THE MAIN THEOREM

To prove the main theorem, we exhibit a point of non-local connectivity, namely,  $s(+\infty)$ . To do this, we construct a sequence of geodesic rays  $\{u_m\} \subset \partial X$  with  $\lim_{m \rightarrow \infty} u_m = s(+\infty)$ , but for  $m \neq n$ ,  $u_m$  and  $u_n$  are in different path components of the complement of the closed set  $L(Ax_0)$  in  $\partial X$ . The theory of Peano spaces will then show  $\partial X$  is not locally connected. (See Theorem 31.4 [W].)

*Proof.* Select an integer  $t_1 \in [0, \infty)$  such that  $t_1 \geq \lambda(2M + 1 + \epsilon)$ . Pick  $\bar{t} > t_1 + \lambda(E + M + Q + \epsilon)$ . By lemma 4.3,  $y = qr_0(\bar{t}) \notin N_{E+M+Q}(r_0(t_1)Ax_0)$ . Let  $u_n : [0, \infty) \rightarrow X$  be a geodesic ray from  $x_0$  passing within  $E$  of  $s^n y \equiv qr_n(\bar{t})$ . Say  $d_X(u_n(\hat{t}_n), s^n y) \leq E$ . See Figure 6. If  $D = d(x_0, y)$  then  $d(x_n, u_n(\hat{t}_n)) \leq d(x_n, s^n y) + d(s^n y, u_n(\hat{t}_n)) \leq D + E$  for all  $n$ . Hence the sequence  $\{u_n\}$  converges to  $s(+\infty) \notin L(Ax_0)$ . We have for large  $m$ ,  $u_m \notin L(Ax_0)$ .

For  $m, n > 0$ , let  $\delta_{m,n}$  be the edge path in  $\Gamma$  from  $s^m$  to  $s^n$  with edge labels defined the obvious reduced word for  $s^{n-m}$ . (See Figure 4 for the five possible cases.) By lemma 4.3, the definition of  $t_1$  and the fact that  $\langle \delta_{m,n}, r_n \rangle$  is reduced:  
 (\*) Neither the path  $q(\delta_{m,n})$  (from  $x_m$  to  $x_n$ ) nor  $N_M(s^m Ax_0)$  intersect  $N_M(r_n(t_1)Ax_0)$  for all  $m$  (and so  $\cup_{i=0}^{\infty} N_M(s^i Ax_0)$  is a subset of a single component of  $X - N_M(r_n(t_1)Ax_0)$ ).

Let  $\beta_n$  be the geodesic from  $s^n y \equiv qr_n(\bar{t})$  to  $u_n(\hat{t}_n)$  (so  $|\beta_n| \leq E$ ). By the definition of  $y$ ,  $d_X(s^n y, r_n(t_1)Ax_0) > E + M + Q$ , so  $\beta_n$  has image in a single path component of  $X - N_M(r_n(t_1)Ax_0)$  and  $d_X(u_n(\hat{t}_n), r_n(t_1)Ax_0) > M + Q$ . As neither  $x_0$  nor  $s^n y$  are points of  $N_M(r_n(t_1)Ax_0)$ , lemma 4.5 implies that  $x_0$  and  $s^n y$

(and so  $x_0$  and  $u_n(\hat{t}_n)$ ) are in different path components of  $X - N_M(r_n(t_1)Ax_0)$ . Hence  $u_n|_{[0, \hat{t}_n]}$  must intersect  $N_M(r_n(t_1)Ax_0)$ . By lemma 4.7,  $u_n([\hat{t}_n, \infty))$  does not intersect  $N_M(r_n(t_1)Ax_0)$ .

As  $s^n y \in u_n([\hat{t}_n, \infty)) \cup im(\beta_n)$  and  $x_0 \in \cup_{i=0}^{\infty} N_M(s^i Ax_0)$  are in different path components of  $X - N_M(r_n(t_1)Ax_0)$ , we have  $u_n([\hat{t}_n, \infty)) \cup im(\beta_n)$  and  $\cup_{i=0}^{\infty} N_M(s^i Ax_0)$  are in different path components of  $X - N_M(r_n(t_1)Ax_0)$ . But then for all  $m, n$ ,  $im(\beta_n) \cup u_n([\hat{t}_n, \infty))$  and  $im(\beta_m) \cup u_m([\hat{t}_m, \infty))$  do not intersect  $N_M(s^{n-1} Ax_0 \cup s^n Ax_0)$ .

By Lemma 4.5,  $s^n y(\in \beta_n)$  and  $s^m y(\in \beta_m)$  are in different path components of  $X - N_M(s^{n-1} Ax_0 \cup s^n Ax_0)$ , so for  $m \neq n$ , any path from  $u_m([\hat{t}_m, \infty))$  to  $u_n([\hat{t}_n, \infty))$  must intersect  $N_M(s^{n-1} Ax_0 \cup s^n Ax_0) \subset N_M(Ax_n \cup Ax_m)$ . This means that for  $m \neq n$ , any path in  $\partial X$  between  $u_n$  and  $u_m$  must intersect  $L(N_M(Ax_n \cup Ax_m) = L(Ax_0))$ .  $\square$

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