

# DYNAMICS OF THE ACTION OF A CAT(0) GROUP ON THE BOUNDARY

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ABSTRACT. The main theorem here gives a geometric condition on the fixed point sets of two hyperbolic isometries of a  $CAT(0)$  group which guarantees that the subgroup generated by the two elements contains a free subgroup. A result in section 3 shows that an element extends to the identity on the boundary if and only if the element is virtually central in the  $CAT(0)$  group. Finally, the two dimensional case is considered in the last section of the paper and there it is shown that powers of two hyperbolic isometries commute if and only if a geometric condition on the fixed point sets is satisfied.

## INTRODUCTION

If  $G$  is a word hyperbolic group, then the action of  $G$  on  $\partial G$  is a convergence group action [5]. This allows one to prove many interesting facts about the group and the boundary. For example, it can be used to classify the subgroups of  $G$  as finite, virtually  $\mathbb{Z}$ , or *large* (i.e. contains a nonabelian free group)[9], [17]. This action is also crucial in the proof of local connectivity of the boundary for one ended word hyperbolic groups, see [5],[16] for instance. In this paper,  $\Gamma$  is a group of isometries of a  $CAT(0)$  space  $X$  which acts properly discontinuously and cocompactly (i.e. *geometrically*) and we study the dynamics of the group acting on  $\partial X$ . There are two interesting topologies on  $\partial X$  - the visual and the Tits' - both are used here. The action of  $\Gamma$  on  $X$  extends to an action by homeomorphisms on the visual boundary and by isometries on the Tits' boundary. The idea is to use these actions to obtain the correct generalizations of the results which hold in the word hyperbolic setting. One result along these line is given in section 3 - there it is shown that an isometry is virtually central in  $\Gamma$  if and only if it acts as the identity on the boundary.

The starting point for the Main Theorem in is a generalization of a lemma proved by V. Schroeder in [4]. The lemma examines how the boundary behaves under the action of one hyperbolic isometry (see Lemma 4.1). This lemma was proved in the setting of nonpositively curved manifolds, and the proof is slightly more delicate in the more general setting of  $CAT(0)$  spaces. In a word hyperbolic group, a hyperbolic isometry has exactly two fixed points, one attracting and one repelling. In the  $CAT(0)$  setting, the fixed point set is much larger for a general hyperbolic isometry and there can be a strictly larger set which is not fixed pointwise, but is

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fixed setwise. There are two special fixed points which can act as an attracting-repelling fixed point pair for points in the boundary which are far away (in the Tits' topology) from the fixed point set.

The following question was posed by D. Wise and is listed on the open problem list in Geometric Group Theory on M. Bestvina's web page.

**Question.** *Suppose  $\Gamma$  acts geometrically on the  $CAT(0)$  space  $X$ . If  $a, b \in \Gamma$  are hyperbolic isometries, does there exist an  $N \in \mathbb{N}$  so that either  $a^N$  and  $b^N$  generate a free subgroup or  $[a^N, b^N] = 1$ ?*

The main theorem here does not answer this question, but rather considers the following related question:

**Question.** *Suppose  $\Gamma$  acts geometrically on the  $CAT(0)$  space  $X$ . If  $a, b \in \Gamma$  are hyperbolic isometries, is there a geometric condition on the action of  $a$  and  $b$  on  $\partial X$  which guarantees that  $a^N$  and  $b^N$  generate a free subgroup or  $[a^N, b^N] = 1$  for some  $N$ ?*

The reason for this more specific question is because of the situation in the word hyperbolic setting. If  $a, b$  are infinite order elements in a word hyperbolic group  $G$ , then the fixed point sets of  $a$  and  $b$  acting on  $\partial G$  completely determine the answer to this question. Indeed, if the fixed point sets intersect, then the subgroup generated by  $a$  and  $b$  contains  $\mathbb{Z}$  as a subgroup of finite index otherwise high enough powers of  $a$  and  $b$  generate a free subgroup.

The main theorem in this paper gives a geometric condition which guarantees that the subgroup generated by  $a$  and  $b$  contains a free subgroup. The terminology *rank one* is defined in section 3. In [3], the authors classify certain 2 dimensional  $CAT(0)$  complexes which admit geometric group actions and one of the types of groups which arises are those which contain rank one elements. The authors give a beautiful proof of the fact that a rank one element can be detected in any Cayley graph of the group.

**Main Theorem.** *If  $a, b \in \Gamma$  are infinite order elements with  $Td(\{a^{\pm\infty}\}, \{b^{\pm\infty}\}) > \pi$ , then the subgroup generated by  $a$  and  $b$  contains a free subgroup. In fact, there exists an  $N > 0$  such that for all  $n \geq N$ ,  $a^n b^{-n}$  is rank one.*

The last section of the paper contains some remarks about when powers of  $a$  and  $b$  commute in the case where the space  $X$  has no flats of dimension higher than 2. The main result there strongly uses a theorem of B. Leeb which can be found in [11]. In the following statement, the notation  $S_a^0$  denotes the 0-sphere  $\{a^{\pm\infty}\}$  which consists of the two endpoints of an axis for the hyperbolic isometry  $a$  and  $\star$  denotes the spherical join.

**Theorem 7.3.** *Suppose  $\Gamma$  is a group acting geometrically on the rank 2  $CAT(0)$  space  $X$  and suppose  $a, b \in \Gamma$  are infinite order elements. Then  $[a^p, b^q] = 1$  for  $p, q > 0$  if and only if there exists an  $S^1$  isometrically embedded in  $(\partial X, Td)$  with image in  $Fix(\bar{a}) \cap Fix(\bar{b})$  which contains  $S_a^0$  and  $S_b^0$  as antipodal pairs.*

The author would like to thank Eric Swenson for pointing out an error in the original proof of Lemma 4.1.

1. DEFINITIONS AND BASIC FACTS

In this section we give definitions and basic properties of CAT(0) spaces, boundaries and isometries as well as some known facts we will need in the proof of the main result. References for this material are [7],[2].

**1.1 CAT(0) Spaces and Boundaries.**

Let  $(X, d)$  be a metric space. Then  $X$  is *proper* if metric balls are compact. A (*unit speed*)*geodesic* from  $x$  to  $y$  for  $x, y \in X$  is a map  $c : [0, D] \rightarrow X$  such that  $c(0) = x$ ,  $c(D) = y$  and  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, D]$ . If  $I \subseteq \mathbb{R}$  then a map  $c : I \rightarrow X$  parametrizes its image *proportional to arclength* if there exists a constant  $\lambda$  such that  $d(c(t), c(t')) = \lambda|t - t'|$  for all  $t, t' \in I$ . Lastly,  $(X, d)$  is called a *geodesic metric space* if every pair of points are joined by a geodesic.

**Definition.** Let  $(X, d)$  be a proper complete geodesic metric space. If  $\Delta abc$  is a geodesic triangle in  $X$ , then we consider  $\Delta \bar{a}\bar{b}\bar{c}$  in  $\mathbb{E}^2$ , a triangle with the same side lengths, and call this a *comparison triangle*. Then we say  $X$  satisfies the CAT(0) inequality if given  $\Delta abc$  in  $X$ , then for any comparison triangle and any two points  $p, q$  on  $\Delta abc$ , the corresponding points  $\bar{p}, \bar{q}$  on the comparison triangle satisfy

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

If  $(X, d)$  is a CAT(0) space, then the following basic properties hold:

- (1) The distance function  $d: X \times X \rightarrow \mathbb{R}$  is convex.
- (2)  $X$  has unique geodesic segments between points.
- (3)  $X$  is contractible.

For details, see [15].

Let  $(X, d)$  be a proper CAT(0) space. First, define the boundary,  $\partial X$  as a set as follows:

**Definition.** Two geodesic rays  $c, c': [0, \infty) \rightarrow X$  are said to be *asymptotic* if there exists a constant  $K$  such that  $d(c(t), c'(t)) \leq K, \forall t > 0$  - this is an equivalence relation. The boundary of  $X$ , denoted  $\partial X$ , is then the set of equivalence classes of geodesic rays. The union  $X \cup \partial X$  will be denoted  $\bar{X}$ . The equivalence class of a ray  $c$  is denoted by  $c(\infty)$ .

There is a natural neighborhood basis for a point in  $\partial X$ . Let  $c$  be a geodesic ray emanating from  $x_0$  and  $r > 0, \epsilon > 0$ . Also, let  $S(x_0, r)$  denote the sphere of radius  $r$  centered at  $x_0$  with  $p_r : X \rightarrow S(x_0, r)$  denoting projection. Define

$$U(c, r, \epsilon) = \{x \in \bar{X} | d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon\}$$

This consists of all points in  $\bar{X}$  such that when projected back to  $S(x_0, r)$ , this projection is not more than  $\epsilon$  away from the intersection of that sphere with  $c$ . These sets along with the metric balls about  $x_0$  form a basis for the *cone topology*. The set  $\partial X$  with the cone topology is often called the *visual boundary*. As one expects, the visual boundary of  $\mathbb{R}^n$  is  $S^{n-1}$  as is the visual boundary of  $\mathbb{H}^n$ . Thus the visual boundary does not capture the difference between these two CAT(0) spaces. In the next section, the Tits' topology on  $\partial X$  is introduced which does distinguish these two boundaries. The notation  $\partial_\infty X$  is used to denote the visual topology and  $\partial_{\text{Tits}} X$  to denote the Tits' boundary.

## 1.2 Isometries.

**Definition.** Let  $\gamma$  be an isometry of the metric space  $X$ . The displacement function  $d_\gamma: X \rightarrow \mathbb{R}_+$  defined by  $d_\gamma(x) = d(\gamma \cdot x, x)$ . The translation length of  $\gamma$  is the number  $|\gamma| = \inf\{d_\gamma(x) : x \in X\}$ . The set of points where  $\gamma$  attains this infimum will be denoted  $\text{Min}(\gamma)$ . An isometry  $\gamma$  is called semi-simple if  $\text{Min}(\gamma)$  is non-empty.

We summarize some basic properties about this  $\text{Min}(\gamma)$  in the following proposition.

**Proposition 1.2.1.** Let  $X$  be a metric space and  $\gamma$  an isometry of  $X$ .

- (1)  $\text{Min}(\gamma)$  is  $\gamma$ -invariant.
- (2) If  $\alpha$  is another isometry of  $X$ , then  $|\alpha\gamma\alpha^{-1}| = |\gamma|$ , and  $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$ ; in particular, if  $\alpha$  commutes with  $\gamma$ , then it leaves  $\text{Min}(\gamma)$  invariant.
- (3) If  $X$  is  $\text{CAT}(0)$ , then the displacement function  $d_\gamma$  is convex: hence  $\text{Min}(\gamma)$  is a closed convex subset of  $X$ .

**Definition.** Let  $X$  be a metric space. An isometry  $\gamma$  of  $X$  is called

- (1) elliptic if  $\gamma$  has a fixed point - i.e  $|\gamma| = 0$  and  $\text{Min}(\gamma)$  is non-empty.
- (2) hyperbolic if  $d_\gamma$  attains a strictly positive infimum.
- (3) parabolic if  $d_\gamma$  does not attain its infimum, in other words if  $\text{Min}(\gamma)$  is empty.

It is clear that an isometry is semi-simple if and only if it is elliptic or hyperbolic. If two isometries are conjugate in  $\text{Isom}(X)$ , then they are in the same class.

When a group  $\Gamma$  acts geometrically on a  $\text{CAT}(0)$  space  $X$ , then the elements of  $\Gamma$  act as semi-simple isometries because of the cocompactness of the action. Our main concern here will be the hyperbolic (or infinite order) isometries of  $\Gamma$ .

Recall that a geodesic line in  $X$  is a map  $c: \mathbb{R} \rightarrow X$  such that  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in \mathbb{R}$ . Two such lines  $c, c'$  are asymptotic if there exists a constant  $K$  such that  $d(c(t), c'(t)) \leq K$  for all  $t \in \mathbb{R}$ . Two lines are parallel if they cobound a flat strip in  $X$ . The following rigidity theorem shows that in a  $\text{CAT}(0)$  space, asymptotic geodesic lines are in fact parallel.

**The Flat Strip Theorem 1.2.2.** Let  $X$  be a  $\text{CAT}(0)$  space and let  $c: \mathbb{R} \rightarrow X$  and  $c': \mathbb{R} \rightarrow X$  be geodesic lines in  $X$ . If  $c$  and  $c'$  are asymptotic, then the convex hull of  $c(\mathbb{R}) \cup c'(\mathbb{R})$  is isometric to a flat strip  $BbbR \times [0, D] \subset \mathbb{E}^2$ .

The next theorem is a structure theorem for  $\text{Min}(\gamma)$  where  $\gamma$  is a hyperbolic isometry of a  $\text{CAT}(0)$  space  $X$ . Proofs of this can be found in [7],[6].

**Theorem 1.2.3.** Let  $X$  be a  $\text{CAT}(0)$  space.

- (1) An isometry  $\gamma$  of  $X$  is hyperbolic if and only if there exists a geodesic line  $c: \mathbb{R} \rightarrow X$  which is translated non-trivially by  $\gamma$ , namely  $\gamma \cdot c(t) = c(t + a)$ , for some  $a > 0$ . The set  $c(\mathbb{R})$  is called an axis of  $\gamma$ . For any such axis, the number  $a$  is equal to  $|\gamma|$ .
- (2) If  $\gamma$  is hyperbolic, the axes of  $\gamma$  are all parallel to each other, and their union is  $\text{Min}(\gamma)$ .
- (3)  $\text{Min}(\gamma)$  is isometric to a product  $Y \times \mathbb{R}$ , and the restriction of  $\gamma$  to  $\text{Min}(\gamma)$  is of the form  $(y, t) \mapsto (y, t + |\gamma|)$ , where  $y \in Y$ ,  $t \in \mathbb{R}$ .
- (4) Every isometry  $\alpha$  which commutes with  $\gamma$  leaves  $\text{Min}(\gamma) = Y \times \mathbb{R}$  invariant, and its restriction to  $Y \times \mathbb{R}$  is of the form  $(\alpha_Y, \alpha_t)$ , where  $\alpha_Y$  is an isometry of  $Y$  and  $\alpha_t$  a translation of  $\mathbb{R}$ .

Another application of the Flat Strip Theorem is the following generalization of Theorem 1.2.3. Proofs can be found in [7] in the CAT(0) setting and [4] in the classical setting.

Recall that if  $A$  is an abelian group then the rank  $rk_{\mathbb{Q}}A$  is the integer  $n$  such that  $A$  modulo its torsion subgroup is isomorphic to  $\mathbb{Z}^n$ .

**Flat Torus Theorem 1.2.4.** *Let  $\Gamma$  be a finitely generated abelian group acting properly by semi-simple isometries on a complete CAT(0) space  $X$ .*

- (1)  $Min(\Gamma) = \bigcap_{\gamma \in \Gamma} Min(\gamma)$  is non-empty and splits as a product  $Y \times \mathbb{E}^n$ , where  $n = rk_{\mathbb{Q}}\Gamma$ ;
- (2) every  $\gamma \in \Gamma$  leaves  $Min(\Gamma)$  invariant, respecting the product structure;  $\gamma$  is the identity on the first factor  $Y$  and a translation on the second factor  $\mathbb{E}^n$ ;
- (3) the quotient of each  $n$ -flat  $\{y\} \times \mathbb{E}^n$  by this action is an  $n$ -torus.

**Corollary 1.2.5.** *If  $\Gamma$  is a group acting geometrically on a CAT(0) space  $X$ , and  $H \cong \mathbb{Z}^n$  a subgroup of  $\Gamma$ , then there exists a point  $x \in X$  such that the orbit  $H \cdot x$  is a lattice in an isometrically embedded copy of  $\mathbb{E}^n$  (i.e. an  $n$ -flat).*

## 2. ANGLES AND THE TITS' METRIC ON $\partial X$

In this section, we develop a general technique for measuring angles between points in  $\partial X$ . We assume the reader has some knowledge of how to measure angles in a metric space, but we add the necessary definitions for completeness. Alexandrov used the method of comparison triangles to define the notion of angle between two geodesics leaving a point  $x_0$  in a metric space  $X$ , [1]. We recall that definition here.

**Definition.** *Let  $c : [0, a] \rightarrow X$  and  $c' : [0, a'] \rightarrow X$  be two geodesics with  $c(0) = c'(0) = x_0$ . Given  $t \in [0, a]$ ,  $t' \in [0, a']$ , and let  $\alpha_{c,c'}^{t,t'}$  denote the angle in a comparison triangle in Euclidean space at the vertex corresponding to  $x_0$ . The (upper) angle between  $c, c'$  at  $x_0$  is defined to be the following number:*

$$\angle_{c,c'} := \limsup_{t,t' \rightarrow 0} \alpha_{c,c'}^{t,t'}$$

Note: The lim sup is used because the limit may not always exist, but in CAT(0) spaces, the limit does exist and instead of calling it an “upper” angle, we call it the angle. For a proof of this, we reference [7].

**Definition.** *Let  $X$  be a CAT(0) space. Given  $x \in X$  and  $u, v \in \partial X$ , we denote by  $\angle_x(u, v)$  the angle between the unique geodesic rays which issue from  $x$  and lie in the classes  $u$  and  $v$  respectively. Then we define the angle between  $u$  and  $v$  to be*

$$\angle(u, v) = \sup_{x \in X} \angle_x(u, v)$$

The following proposition gives some basic facts about angles between boundary points of a CAT(0) space (need completeness here which is assumed for all spaces here). The proofs of these involve repeated use of the CAT(0) inequality and the convexity of the distance function. For complete proofs, see chapter 2 of [7].

**Proposition 2.1.** *Let  $X$  be a  $CAT(0)$  space and let  $c, c'$  be two geodesic rays issuing from the same point  $x \in X$ . Let  $u = c(\infty)$  and  $u' = c'(\infty)$ .*

- (1) *If  $u \neq u'$ , then there exists  $t > 0$  such that  $\angle_{c(t)}(u, u') > 0$ ; hence  $\angle(u, u') > 0$ .*
- (2) *The function  $y \mapsto \angle_y(u, u')$  is upper semicontinuous on  $X$ .*
- (3) *The function  $t \mapsto \angle_{c(t)}(u, u')$  is non-decreasing and*

$$\angle(u, u') = \lim_{t \rightarrow \infty} \angle_{c(t)}(u, u')$$

- (4) *If  $\angle_x(u, u') = \angle(u, u')$ , then the convex hull of  $c(\mathbb{R}_+) \cup c'(\mathbb{R}_+)$  is isometric to a sector in the Euclidean plane bounded by two rays which meet at an angle  $\angle(u, u')$ .*
- (5) *The function  $\angle : \partial X \times \partial X \rightarrow [0, \pi]$  is lower semicontinuous.*
- (6)  $\lim_{t \rightarrow \infty} \frac{d(c(t), c'(t))}{t} = [2 - 2 \cos(\angle(u, u'))]^{\frac{1}{2}}$

It is easy to see that this angle satisfies the triangle inequality and so this combined with (1) from above gives us that  $\angle(u, u')$  defines a metric on  $\partial X$  called the *angular metric*.

**Example.** Consider the  $CAT(0)$  space  $X = \mathbb{H}^2$ . We know  $\partial X$  is the unit circle. For any two points on the boundary circle, there is a geodesic line in  $\mathbb{H}^2$  joining these two points. Because the above calculation is independent of basepoint, we may as well assume the basepoint is on this line. Then one easily sees that the angle between the two points must be  $\pi$ . This means the angle metric on  $\partial(\mathbb{H}^2)$  is discrete.

The following fundamental theorem about the angular metric was proved by Gromov in the case of nonpositively curved manifolds [4] and a proof in the  $CAT(0)$  setting is given in [10]. A  $CAT(1)$  space is defined similarly to a  $CAT(0)$  space except one compares triangles (of diameter less than  $\pi$ ) in the given space to triangles in  $S^2$ .

**Theorem 2.2.** *If  $X$  is a complete  $CAT(0)$  space, then  $\partial X$  with the angular metric is a  $CAT(1)$  space.*

The following proposition is an extremely useful fact concerning triangles with one vertex at infinity and will be used in the proof of Lemma 4.1. A proof can be found in [7].

**Proposition 2.3.** *Let  $\Delta$  be a geodesic triangle in a  $CAT(0)$  space  $X$  with one vertex at infinity; thus  $\Delta$  consists of two asymptotic rays  $c$  and  $c'$ , together with the geodesic segment joining  $c(0)$  and  $c'(0)$ . Let  $x = c(0)$ ,  $x' = c'(0)$  and  $\sigma = c(\infty) = c'(\infty)$ , and let  $\gamma = \angle_x(x', \sigma)$ ,  $\gamma' = \angle_{x'}(x, \sigma)$ . Then,  $\gamma + \gamma' \leq \pi$  with equality if and only if the convex hull of  $\Delta$  is isometric to the convex hull of a geodesic triangle in the Euclidean plane which has one vertex at infinity and has interior angles  $\gamma$  and  $\gamma'$ .*

The *Tits' metric* on  $\partial X$ , denoted  $Td$ , is the *length* metric associated to the angular metric. Thus for  $v, w \in \partial X$ ,  $Td(v, w)$  is defined to be the infimum of the lengths of rectifiable curves in the angular metric between  $v$  and  $w$ . If there are no rectifiable curves joining them, then  $Td(v, w) = \infty$ . This is true for any two points

in  $\partial\mathbb{H}^2$  for instance. The Tits' boundary is denoted  $\partial_{Tits}X$ . In the next section we examine the Tits' boundary on  $\mathbb{H}^2 \times \mathbb{R}$  more closely.

The basic facts needed here concerning the Tits' metric are contained in the following proposition. Again, proofs in the manifold setting can be found in [4] and in [7] for the CAT(0) setting.

**Proposition 2.4.** *Let  $X$  be a (proper) CAT(0) space and let  $v, w$  be distinct points of  $\partial X$ .*

- (1) *If  $Td(v, w) > \pi$ , then there is a geodesic line  $c : \mathbb{R} \rightarrow X$  with  $c(\infty) = v$  and  $c(-\infty) = w$ .*
- (2) *If there is no geodesic line in  $X$  between  $v$  and  $w$ , then  $Td(v, w) = \angle(v, w)$  and there is a geodesic segment in  $\partial_{Tits}X$  joining  $v$  and  $w$ .*
- (3) *If  $c : \mathbb{R} \rightarrow X$  is a geodesic, then  $Td(c(\infty), c(-\infty)) \geq \pi$  with equality if and only if  $c(\mathbb{R})$  bounds a flat half plane in  $X$ .*

### 3. ISOMETRIES WHOSE EXTENSION IS TRIVIAL

In this section we show that if  $\Gamma$  acts geometrically on a CAT(0) space  $X$  then  $\gamma \in \Gamma$  is a hyperbolic isometry whose extension to the boundary is the identity if and only if  $\gamma$  is virtually central. It is worth pointing out here that we do not need the space  $X$  to have local extendability of geodesics to obtain this result. We only need the following weaker notion of *almost extendability*.

**Definition.** *A CAT(0) space  $X$  is almost extendible if there exists a constant  $E$  such that for any pair of points  $x, y$  in  $X$ , there is a geodesic ray  $r : [0, \infty) \rightarrow X$  such that  $r(0) = x$  and  $r$  passes within  $E$  of  $y$ . The number  $E$  is the almost extendibility constant.*

It is a theorem of P. Ontaneda [14] that if  $G$  acts geometrically on a CAT(0) space  $X$ , then  $X$  is almost extendible.

A subset  $M$  of  $X$  is called *quasi-dense* if there exists a constant  $K > 0$  so that each point of  $X$  is within  $K$  of some point of  $M$ . Notice that if a group  $\Gamma$  acts cocompactly, then the orbit of a point is a quasi-dense subset.

**Proposition 3.1.** *If  $M \subset X$  is closed, convex, and quasi-dense, then  $\partial X = \partial M$ .*

*Proof.* We know that  $M$  is CAT(0) in the induced metric and  $\partial M$  embeds in  $\partial X$  so we need only show that this embedding is onto. This follows easily from the following: (this is a lemma in my thesis, but it's easy)

If  $\sigma : \mathbb{R}^+ \rightarrow X$  is a geodesic ray with  $\sigma(t) \in M$  for some  $t \in \mathbb{R}^+$ , then  $\sigma(\mathbb{R}^+) \subset M$ .

If we choose our basepoint to be in  $M$  all rays will have image in  $M$  giving their endpoints in  $\partial M$ .  $\square$

*Remark.* It follows that if  $X$  and  $Y$  are two CAT(0) spaces, then  $X \times Y$  is also CAT(0) (with the product metric) and  $\partial(X \times Y) = \partial X \star \partial Y$  where  $\star$  denotes the spherical join. In fact, this is true in both the visual and Tits' topologies. In this paper the case in which one of the factors is  $\mathbb{R}$  is used so we point out the following:  $\partial(Y \times \mathbb{R}) = \Sigma(\partial Y)$  where  $\Sigma$  denotes suspension. In particular,  $\partial \text{Min}(\gamma) = \Sigma(\partial Y)$  and it is clear that  $\gamma$  acts trivially on this set when extended to a homeomorphism of  $\partial X$ .

**Definition.** Suppose  $\gamma$  is a hyperbolic isometry of the CAT(0) space  $X$ . Consider  $\gamma$  acting as a homeomorphism of the visual boundary  $\partial X$ . Denote by  $\bar{\gamma}$  this homeomorphism and denote by  $\{\gamma^{\pm\infty}\}$  the two fixed points of  $\bar{\gamma}$  which are the suspension points of  $\partial\text{Min}(\gamma)$ .

**Theorem 3.4.** Suppose  $\Gamma$  acts geometrically on the CAT(0) space  $X$ . Then  $\gamma \in \Gamma$  is virtually central in  $\Gamma$  if and only if  $\bar{\gamma}$  is the identity on  $\partial X$ .

*Proof.* The cocompactness is the only point which requires proof. Fix a basepoint  $x_0 \in \text{Min}(\gamma)$ . Let  $K$  be a closed compact fundamental domain for the action of  $\Gamma$  on  $X$ .

Suppose there is no compact set whose  $C(\gamma)$  translates cover  $\text{Min}(\gamma)$ . Then for each  $n > 0$ , choose an  $x_n \in \text{Min}(\gamma)$  such that  $d(x_n, C(\gamma) \cdot x_0) > n$ . Thus  $d(x_n, C(\gamma)) \rightarrow \infty$  as  $n \rightarrow \infty$ .

For each  $n$ , choose  $\beta_n \in \Gamma - C(\gamma)$  such that  $\beta_n \cdot x_n \in K$ . We can assume  $\beta_n \neq \beta_m$  for  $n \neq m$  since

$$d(\beta_n^{-1} \cdot K, K) \geq d(x_0, x_n) - 2\text{diam}(K) \rightarrow \infty \text{ as } n \rightarrow \infty$$

Next we show that the family  $\{d_{\beta_n \gamma \beta_n^{-1}}\}$  of displacement functions is uniformly bounded on  $K$ . Indeed, for any  $y \in K$ , consider the following,

$$\begin{aligned} d_{\beta_n \gamma \beta_n^{-1}}(y) &= d(y, \beta_n \gamma \beta_n^{-1} \cdot y) \\ &\leq d(y, \beta_n \cdot x_n) + d(\beta_n \cdot x_n, \beta_n \gamma \beta_n^{-1}(\beta_n \cdot x_n)) \\ &\quad + d(\beta_n \gamma \beta_n^{-1}(\beta_n \cdot x_n), \beta_n \gamma \beta_n^{-1} \cdot y) \\ &= d(y, \beta_n \cdot x_n) + d(\beta_n \cdot x_n, \beta_n \gamma \cdot x_n) + d(\beta_n \gamma \cdot x_n, \beta_n \gamma \beta_n^{-1} \cdot y) \\ &\leq \text{diam}(K) + |\gamma| + \text{diam}(K) \end{aligned}$$

The last inequality follows since  $x_n \in \text{Min}(\gamma)$ . The action of  $\Gamma$  on  $X$  is proper and  $K$  is compact so we must have  $\beta_n \gamma \beta_n^{-1} = \beta_m \gamma \beta_m^{-1}$  for  $n \neq m$  (by passing to a subsequence if necessary). But this gives  $\beta_m^{-1} \beta_n \in C(\gamma)$  for all pairs  $n \neq m$ .

Now for all  $n \neq 1$  we have the following:

$$\begin{aligned} d(x_n, \beta_n^{-1} \beta_1 \cdot x_0) &\leq d(x_n, \beta_n^{-1} \beta_1 \cdot x_1) + d(\beta_n^{-1} \beta_1 \cdot x_1, \beta_n^{-1} \beta_1 \cdot x_0) \\ &\leq \text{diam}(K) + d(x_1, x_0) \end{aligned}$$

This is a contradiction since  $\beta_n^{-1} \beta_1 \cdot x_0 \in C(\gamma) \cdot x_0$  yet we chose  $x_n$  so that  $d(x_n, C(\gamma) \cdot x_0) \rightarrow \infty$  as  $n \rightarrow \infty$ .  $\square$

Denote by  $\text{Fix}(\bar{\gamma})$  the set of elements of  $\partial X$  which are fixed by the homeomorphism  $\bar{\gamma}$ .

**Theorem 3.3.** With the above assumptions,  $\text{Fix}(\bar{\gamma}) = \partial\text{Min}(\gamma)$ .

*Proof.* Obviously,  $\bar{\gamma}$  acts trivially on  $\partial\text{Min}(\gamma)$ , since  $\gamma$  acts by translation on the set  $\text{Min}(\gamma)$  inside  $X$ . Now suppose  $z \in \text{Fix}(\bar{\gamma})$ . Fix a basepoint  $x_0 \in \text{Min}(\gamma)$  and let  $c$  be a geodesic ray with  $c(0) = x_0$  ending at  $z$ . Then  $\bar{\gamma}(z)$  is the endpoint of  $c(\cdot)$  away based at  $c(\cdot)$  ending at  $z$ . Then the rays  $c$  and  $c(\cdot)$  are asymptotic

rays. We know  $x_0$  was on an axis for  $\gamma$  so  $\gamma \cdot x_0$  is on the same axis but translated by  $|\gamma|$  units. Also,  $\gamma \cdot c(t)$  must lie on  $\gamma(c)$  at the unique point which is  $t$  units away from  $\gamma \cdot x_0$ . Since the rays are asymptotic, we know there exists a constant  $K$  so that  $d(c(t), \gamma \cdot c(t)) \leq K$  for all  $t$ . But this constant must be  $|\gamma|$  since the two rays start out  $|\gamma|$  units apart. Thus  $\gamma \cdot c(t) = \gamma(c)(t)$  which implies  $d(c(t), \gamma \cdot c(t)) = |\gamma|$  giving  $z \in \partial \text{Min}(\gamma)$  as needed.  $\square$

We are now ready to prove the following:

**Theorem 3.4.** *Suppose  $\Gamma$  acts geometrically on the CAT(0) space  $X$ . Then  $\gamma \in \Gamma$  is virtually central in  $\Gamma$  if and only if  $\bar{\gamma}$  is the identity on  $\partial X$ .*

*Proof.* Suppose  $\gamma$  is virtually central in  $\Gamma$ , then there exists a finite index subgroup  $\Gamma'$  in which  $\gamma$  is central. Since  $\Gamma'$  is finite index, it also acts cocompactly on  $X$ . Since  $\gamma$  is central in  $\Gamma'$ ,  $\text{Min}(\gamma)$  is invariant under the action of  $\Gamma'$ . So  $\text{Min}(\gamma)$  is a closed, convex,  $\Gamma'$ -invariant subset of  $X$  - and as such is quasi-dense in  $X$ . By Proposition 3.1,  $\partial X = \partial \text{Min}(\gamma)$  and we know  $\bar{\gamma}$  acts as the identity on  $\partial \text{Min}(\gamma)$ .

If  $\bar{\gamma}$  is the identity on  $\partial X$ , then  $\text{Fix}(\bar{\gamma}) = \partial \text{Min}(\gamma) = \partial X$ . This implies  $\text{Min}(\gamma)$  is quasi-dense in  $X$ . To see this, you must use the fact that  $X$  is almost extendible by [14]. Fix a basepoint  $x_0$  in  $\text{Min}(\gamma)$  and let  $E$  denote the almost extendibility constant. For  $x \in X$  arbitrary, there is a ray beginning at  $x_0$  which passes within  $E$  of  $x$ . This ray has endpoint in  $\partial X = \partial \text{Min}(\gamma)$ . By convexity of  $\text{Min}(\gamma)$ , the ray lies in  $\text{Min}(\gamma)$  and therefore  $x$  lies within  $E$  of  $\text{Min}(\gamma)$ .

Using Theorem 3.2, we see  $C_\Gamma(\gamma)$  is a subgroup of finite index in  $\Gamma$ . Indeed, since the orbit of this centralizer is quasi-dense in  $\text{Min}(\gamma)$  and this set is quasi-dense in  $X$ , we have that the orbit of the centralizer is quasi-dense in  $X$ . But then  $X$  is quasi-isometric to the group  $\Gamma$ . Thus we can find a constant  $K$  so that every  $\gamma \in \Gamma$  lies within  $K$  of an element of the centralizer.

Finally, since  $\gamma$  is central in  $C_\Gamma(\gamma)$ , we have  $\gamma$  virtually central in  $\Gamma$ .

#### 4. PRELIMINARIES

The starting point for the ideas behind the Main Theorem came from the following theorem of Schroeder about the dynamics of a hyperbolic isometry on the boundary given in [4]. The proof given there is for the case of nonpositively curved manifolds and the proof here is a bit more delicate in the more general setting of CAT(0) spaces.

**Lemma 4.1.** *Let  $\gamma$  be a hyperbolic isometry of a CAT(0) space  $X$  and let  $c$  be an axis of  $\gamma$ . Let  $z \in \partial X$ ,  $z \neq \gamma^{-\infty}$  and let  $z_i = \gamma^i \cdot z$ ,  $i \in \mathbb{N}$ . If  $w \in \partial X$  is an accumulation point of the sequence  $\{z_i\}$  (in the cone topology), then  $\angle(\gamma^{-\infty}, w) + \angle(w, \gamma^\infty) = \pi$ . If  $w \neq \gamma^\infty$ , then  $Td(\gamma^{-\infty}, w) + Td(w, \gamma^\infty) = \pi$ . In this case  $c$  and a ray from  $c(0)$  to  $w$  span a flat half plane.*

*Proof.* Let  $\alpha = \angle(\gamma^{-\infty}, z)$ . We first show the following two inequalities hold:

- (1)  $\angle(\gamma^{-\infty}, w) \leq \alpha$
- (2)  $\angle(w, \gamma^\infty) \leq \pi - \alpha$

We use propositions 2.1 and 2.4 several times for properties of angles and the Tits' metric.

The first follows from the lower semi-continuity of the angle metric. To prove the second, suppose the contrary, i.e. that  $\angle(w, \gamma^\infty) > \pi - \alpha$  for some  $\alpha > 0$ . We

show that it is possible to build a triangle with one vertex at infinity which violates Proposition 2.3 to obtain a contradiction.

There exists  $T > 0$  such that for all  $t \geq T$

$$\angle_{c(t)}(w, \gamma^\infty) > \pi + \epsilon_0.$$

For each fixed  $t$ , there exists  $I_t > 0$  such that for all  $i \geq I_t$ ,  $\angle_{c(t)}(z_i, \gamma^\infty) > \pi + \epsilon_0$  which gives

$$\angle_{c(t-i|\gamma|)}(z, \gamma^\infty) > \pi + \epsilon_0.$$

On the other hand, because  $\angle(z, \gamma^{-\infty}) = \alpha$ , given any  $\epsilon > 0$ , there is an  $S > 0$  such that for all  $s \geq S$ ,

$$\angle_{c(-s)}(z, \gamma^{-\infty}) > \alpha - \epsilon.$$

Using  $\epsilon_0$ , choose an appropriate  $S$  as above. Choose  $i \geq I_T$  with  $(T - i|\gamma|) < -S$ . Now the triangle with vertices  $c(-S)$ ,  $c(T - i|\gamma|)$  and  $z$  violates Proposition 2.3 as needed.

By the triangle inequality, we have

$$\pi = \angle(\gamma^{-\infty}, \gamma^\infty) \leq \angle(\gamma^{-\infty}, w) + \angle(w, \gamma^\infty) \leq \pi$$

Thus the inequalities in (1) and (2) are in fact, equalities.

If  $w \neq \gamma^\infty$ , then  $\angle(w, \gamma^{-\infty}) = \alpha = Td(w, \gamma^{-\infty})$  and  $\angle(w, \gamma^\infty) = \pi - \alpha = Td(w, \gamma^\infty)$  and thus the last statement follows from Proposition 2.4(3).  $\square$

*Remark.* If  $\angle(z, \gamma^{-\infty}) = \pi$ , then  $\{z_i\}$  converges to  $\gamma^\infty$ . In particular, if there is a geodesic line in  $X$  from  $z$  to  $\gamma^{-\infty}$ , then  $\{z_i\}$  converges to  $\gamma^\infty$ .

If  $\gamma$  is a hyperbolic isometry of  $X$  and  $c$  is an axis for  $\gamma$ , let  $P_\gamma$  denote the set of all geodesic lines in  $X$  which are parallel to  $c$ . Thus  $\text{Min}(\gamma) \subset P_\gamma$ , but in general  $P_\gamma$  is larger as the following example shows.

**Example 4.2.** Let  $X = \mathbb{R}^2$  and  $\Gamma$  the fundamental group of the Klein bottle with generators  $a$  and  $b$ . If  $a$  is a glide reflection with axis  $c$ , then  $\text{Min}(a) = \{c(0)\} \times \mathbb{R}$  whereas  $P_a = X$ .

$P_\gamma = Y \times \mathbb{R}$  where the endpoints of the  $\mathbb{R}$  factor are  $\{\gamma^{\pm\infty}\}$ . Thus  $\bar{\gamma}$  leaves  $\partial P_\gamma$  set-wise fixed. Theorem 4.1 implies that  $\partial P_\gamma$  is an attracting set for the action of  $\bar{\gamma}$ . In case  $\Gamma$  is word hyperbolic, this set is always  $\{\gamma^{\pm\infty}\}$  with  $\gamma^\infty$  an attracting fixed point and  $\gamma^{-\infty}$  a repelling fixed point. Here the situation is much more complicated. We conclude this section with an example to illustrate what can happen in this more general setting.

If  $X = Y \times \mathbb{R}$ , we obtain  $\partial X \equiv \Sigma(\partial Y)$  (see the remark following Proposition 3.1 for information on products). We obtain the suspension as follows: Glue  $\partial Y \times [0, \frac{\pi}{2}]$  and  $\partial Y \times [-\frac{\pi}{2}, 0]$  together where the 0-levels of both copies are identified via the identity map on  $\partial Y$  and the  $\pm\frac{\pi}{2}$ -levels in each are shrunk to a point. We denote the equivalence class of a point by  $z = [z', \theta_z]$ . With this notation,  $\theta_z = 0$  means  $z = z' \in \partial Y$  and  $\theta_z = \pm\frac{\pi}{2}$  means  $z$  is one of the suspension points.

**Example 4.3.** Consider the CAT(0) space  $X = \mathbb{H}^2 \times \mathbb{R}$  with the group  $\Gamma = F \times \mathbb{Z}$  acting on  $X$  where  $F$  is a cocompact fuchsian group. Consider  $\gamma = (f, n)$  with  $f \neq 1_F$  acting on  $X$ . Thus  $\text{Min}(\gamma) = \mathbb{R} \times \mathbb{R}$  where the first factor is the axis for  $f$  inside  $\mathbb{H}^2$ . Note that  $\partial X = \Sigma(\partial Y)$  while  $\partial Y$  is the suspension of a diameter

metric on  $S^1$ . In particular,  $\partial_{Tits}X$  is not even locally compact at the suspension points.

Let  $z \in \partial X$  be a point which is not in  $\text{Fix}(\gamma) = \partial\text{Min}(\gamma)$ . Thus  $z = [z', \theta_z]$  for some  $z' \in S^1$  and  $\theta_z \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Without loss of generality, assume  $0 \leq \theta_z \leq \frac{\pi}{2}$ . Also,  $\gamma^\infty = [f^\infty, \theta_\gamma]$  and  $\gamma^{-\infty} = [f^{-\infty}, -\theta_\gamma]$  where  $\theta_\gamma = \arctan(\frac{n}{|\gamma|})$ ; we can also assume  $0 \leq \theta_\gamma \leq \frac{\pi}{2}$ . Clearly  $\angle(\gamma^{-\infty}, z) = \pi - |\theta_z - \theta_\gamma|$  and  $\angle(\gamma^\infty, z) = \pi - (\theta_z + \theta_\gamma)$ , thus we have

$$\lim_{i \rightarrow \infty} \gamma^i \cdot z = [f^\infty, \theta_z]$$

$$\lim_{i \rightarrow \infty} \gamma^{-i} \cdot z = [f^{-\infty}, \theta_z].$$

Notice if  $\theta_z = \theta_\gamma$ , then  $z$  limits to  $\gamma^\infty$  under the action of  $\gamma$ . However,  $z$  does not limit to  $\gamma^{-\infty}$  under the action of  $\gamma^{-1}$ .

## 5. RANK ONE ISOMETRIES

In the paper [3], the authors introduce the notion of a *rank one* element in a CAT(0) group  $\Gamma$ . This notion is used in the proof of the Main Theorem so several definitions and lemmas from [3] are quoted without proof in this section.

**Definition.** *Suppose  $\gamma$  is a hyperbolic isometry of a CAT(0) space  $X$ . Then  $\gamma$  is called rank one if no axis of  $\gamma$  bounds a flat half plane in  $X$ . A CAT(0) group is called rank one if it contains a rank one element.*

**Lemma 5.1.** *Let  $\sigma : \mathbb{R} \rightarrow X$  be a unit speed geodesic which does not bound a flat strip of width  $R > 0$ . Then there are neighborhoods  $U$  of  $\sigma(-\infty)$  and  $V$  of  $\sigma(\infty)$  in  $\overline{X}$  such that for any  $\psi \in U$  and  $\nu \in V$  there is a geodesic line from  $\psi$  to  $\nu$ , and for any such geodesic  $\sigma'$  we have  $d(\sigma', \sigma(0)) < R$ . Moreover,  $\sigma'$  does not bound a flat strip of width  $2R$ .*

The main property of a rank one element is contained in the following lemma which can be found in [2] in the manifold case and in [3] for the more general setting. The main theorem in [3] says a bit more than we say here, but this is the statement needed here. The lemma shows that the dynamics of the action of a rank one isometry on the boundary are exactly the same as in the setting of negative curvature - namely, there are exactly two fixed points, one attracting and one repelling.

**Lemma 5.2.** *Suppose  $\gamma$  is a rank one isometry of the CAT(0) space  $X$ . Given neighborhoods  $U$  of  $\gamma^\infty$  and  $V$  of  $\gamma^{-\infty}$ , there exists an  $N \in \mathbb{N}$  such that  $\overline{\gamma^n}(\partial X - U) \subset V$  and  $\overline{\gamma^{-n}}(\partial X - V) \subset U$ .*

**Lemma 5.3.** *Let  $\sigma : \mathbb{R} \rightarrow X$  be a unit speed geodesic which does not bound a flat half plane. Let  $(\phi_n)$  be a sequence of isometries of  $X$  such that  $\phi_n(x) \rightarrow \sigma(\infty)$  and  $\phi_n^{-1}(x) \rightarrow \sigma(-\infty)$  for one (and hence any)  $x \in X$ . Then for sufficiently large  $n$ ,  $\phi_n$  has an axis  $\sigma_n$  such that  $\sigma_n(\infty) \rightarrow \sigma(\infty)$  and  $\sigma_n(-\infty) \rightarrow \sigma(-\infty)$  and  $n \rightarrow \infty$ .*

The following theorem allows us to construct the necessary free group in the main theorem here. This theorem is proved in both [3] and [2] but in slightly different forms. We give an outline of proof here for completeness.

**Theorem 5.4** - see **Theorem 4.6** in [2]. *Suppose  $H$  is a group of hyperbolic isometries acting properly discontinuously on a  $CAT(0)$  space  $X$  and suppose  $H$  contains a rank 1 element. Then  $H$  is either virtually  $\mathbb{Z}$  or  $H$  contains a nonabelian free group.*

*Outline of Proof.* Suppose  $\gamma$  is a rank one element in  $H$  and that one (and hence any) axis for  $\gamma$  does not bound a flat half strip of width  $D \geq 0$ .

Let  $c$  be an axis for  $\gamma$ . If  $H$  is not virtually  $\mathbb{Z}$ , then there exists an element  $\alpha \in H$  such that  $\bar{\alpha}$  moves one of the endpoints  $\gamma^{\pm\infty}$ . In fact, we must have  $(\alpha\gamma\alpha^{-1})^{\pm\infty} \cap \gamma^{\pm\infty} = \emptyset$ . Indeed, with a proper group action, two hyperbolic isometries cannot have just one endpoint in common. This is the contents of lemma 4.5 in [3] but follows quite easily from the properness of the action. Now we have  $\alpha \cdot c$  is an axis for  $\alpha\gamma\alpha^{-1}$  and this axis does not bound a flat half strip of width  $D$  - i.e. this element is also rank 1.

Now choose 4 disjoint neighborhoods around these four points and choose  $N_1$  and  $N_2$  as in lemma 3.6 for  $\gamma$  and  $\alpha\gamma\alpha^{-1}$  respectively. If  $N$  is the larger of  $N_1$  and  $N_2$ , then one can show  $\gamma^N$  and  $(\alpha\gamma\alpha^{-1})^N$  generate a free subgroup by showing that every nontrivial reduced word in these elements must move some point of the boundary thereby implying it cannot be the identity. For more details, see Theorem 3.1 in [2] or Theorem 4.6 in [3].  $\square$

## 6. MAIN THEOREM

In this section we give the proof of the Main Theorem along with several lemmas which are useful on their own. The first of these is not used explicitly in the proof of the Main Theorem, however this lemma motivates why the condition of having Tits distance larger than  $\pi$  is an important condition to consider.

**Lemma 6.1.** *Suppose  $C_1$  and  $C_2$  are disjoint closed geodesically convex subsets of  $X$  with  $Td(\partial C_1, \partial C_2) > \pi$ . Let  $x_i \in C_i$  such that  $d(x_1, x_2) = d(C_1, C_2)$ . There exists a constant  $K > 0$  such that for all  $z_i \in \partial C_i$   $i = 1, 2$ , the geodesic line in  $X$  between them lies within  $K$  of  $[x_1, x_2]$ .*

*Proof.* Let  $z \in \partial C_1$ ,  $w \in \partial C_2$ . Since  $Td(\partial C_1, \partial C_2) > \pi$ , there exists a geodesic line  $\sigma_{zw} : \mathbb{R} \rightarrow X$  with  $\sigma(-\infty) = w$ ,  $\sigma(\infty) = z$  and this line does not bound a flat strip of width  $R > 0$  by Proposition 2.3.

Fix a  $z \in \partial C_1$  and for each  $w \in \partial C_2$ , choose open sets  $U_w$  and  $V_w$  as Lemma 5.1. The sets  $V_w$  form an open cover of the compact set  $\partial C_2$ . Choose a finite subcover  $V_{w_1}, \dots, V_{w_m}$ . Let  $U_z = \cap \{U_w : w \in \partial C_2\}$ .

For each  $i = 1, \dots, m$  there exists a  $K > 0$  such that the geodesic line  $\sigma_{zw}$  where  $w$  is any point of  $V_{w_i}$  lies within  $K$  of the line  $\sigma_{zw_i}$  by Lemma 5.1.

Now cover  $\partial C_2$  by the sets  $U_z$  and choose a finite subcover  $U_{z_1}, \dots, U_{z_n}$ . For each  $j = 1, \dots, n$  we have an associated finite subcover of  $\partial C_2$  each with a constant  $K$  as above. Work with the union of these covers of  $\partial C_2$  and choose the maximum of all the finitely many possible  $K$ 's - call this number  $K$  also.

Let  $z \in \partial C_1$  and  $w \in \partial C_2$  be arbitrary.  $z \in U_{z_i}$  for some  $i$  and  $w \in V_{w_j}$  for some  $j$  so that  $\sigma_{zw}$  lies within  $K$  of  $\sigma_{z_i w_j}$ . Thus to find  $d(\sigma_{zw}, [x_1, x_2])$ , it suffices to check  $d(\sigma_{z_i w_j}, [x_1, x_2])$ . Since there are only finitely many of these, we are done.  $\square$

*Remark.* Observe that  $C_1 \cap C_2$  is either empty or bounded. If it were unbounded it would contain a ray which had endpoints in  $\partial C_1 \cap \partial C_2$  which contradicts the assumption.

tion  $Td(\partial C_1, \partial C_2) > \pi$ . Thus the theorem holds if the intersection is nonempty, simply replace  $[x_1, x_2]$  in the conclusion with the center of the (bounded) intersection.

The following example is a  $CAT(0)$  group which contains a rank one isometry but also contains isometries which are not rank one. This cannot happen in the manifold setting - specifically, if a 2-dimensional nonpositively curved manifold contains a rank one element, then all elements are rank one and the fundamental group is word hyperbolic, [8]. We point out sets which satisfy the hypotheses of Lemma 6.1 and also give some indication of how the generators act on  $\partial X$ . This example is also used in [13] to illustrate how certain HNN extensions acting on a  $CAT(0)$  space force the boundary of the space to contain points of non-local connectivity.

**Example 6.2.** Let  $G = A \star_{C_1=C_2}$  where  $A = \mathbb{Z} \oplus \mathbb{Z}$  with generators  $x, y$ ,  $C_1 = \langle x \rangle$ ,  $C_2 = \langle y \rangle$  and the stable letter  $z$  identifies  $C_1$  and  $C_2$  via the relation  $z^{-1}xz = y$ . Thus  $G$  has presentation  $G = \langle x, y, z \mid [x, y] = 1, z^{-1}xz = y \rangle$ .  $G$  acts on a  $CAT(0)$  space  $X$  which can be obtained as the universal cover of the mapping torus for a certain homeomorphism of a once punctured torus (this can be seen by viewing  $G$  as a semi-direct product of  $F_2$  and  $\mathbb{Z}$ ). We describe  $X$  explicitly here.

$X$  can be viewed as a collection of planes (copies of the universal cover of the original torus) glued together along lines with strips inbetween them via the identifications given by the HNN description of the group  $G$ . Start with a plane  $P_0$  whose origin will be the basepoint of  $X$  so that  $P_0$  contains the orbit of the  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $x$  and  $y$ . Label the edges of this plane with  $x$  and  $y$  labels (we can assume that  $x$  and  $y$  act as translations of length 1 in perpendicular directions inside  $P_0$ ). Glue a vertical strip of height one along the  $x$  axis of  $P_0$  with each of the vertical edges labeled by a  $z$ . Because of the identifications in  $G$ , the top edge of this strip is labeled by  $y$  edges. These  $y$  edges lie in a plane, denoted  $P_1$  which is the image of  $P_0$  under the action of  $z$ . Thus we have glued together two planes and a strip. Notice that the  $x$ -axis in  $P_0$  is parallel to the  $y$ -axis in  $P_1$ . Denote by  $P_n$  the plane which is the image of  $P_0$  under the action of  $z^n$ .

In  $P_0$ , denote the  $x$ -axis by  $A$ . Since  $y$  commutes with  $x$ ,  $yA$  is another axis for  $x$  which is disjoint and parallel to  $A$ . Thus we have a family of parallel lines in  $P_0$  given by  $y^n A$ . To each of these lines, we glue a plane as above. The plane we glue with a strip along  $y^n A$  is  $y^n P_1$ . This plane is totally disjoint from the plane  $P_1$ , but they share a common direction.

Iterate this process everywhere according to the group  $G$  - i.e. in each plane there is an  $x$ -axis (actually the image of  $A$  under some group element) so we glue another strip with a plane on top where the top of the strip has  $y$  edge labels in the new plane and the vertical edges in the strip are labeled by  $z$ .

This is not all of  $X$  because we have not considered the action of negative powers of  $z$  on the planes constructed. There is another "half" of  $X$  which is constructed via the relation  $zyz^{-1} = x$  which lies below the part described above.

Each plane contributes a circle to  $\partial X$ . For  $P_n$ , denote the boundary circle by  $S_n$ . Because  $P_0$  and  $P_1$  share a common direction,  $S_0$  and  $S_1$  have two points in common - likewise for  $S_n$  and  $S_{n+1}$ . In  $P_0$ , the endpoints of  $A$ , denoted  $x^{\pm\infty}$ , are the same as the endpoints for the  $y$ -axis in  $P_1$ . These are labeled by  $zy^{\pm\infty}$  since this line is obtained as the image of the  $y$ -axis in  $P_0$  under the action of  $z$ . These are the only two points in  $S_0 \cap S_1$ . The points  $x^{\pm\infty}$  are also in  $S_1$  but these are

the same as  $z^2 y^{\pm\infty}$  in  $S_2$ . Iterate this process under the action of  $z$  on  $\partial X$ , to get a collection of circles which limits to a single point, namely  $z^\infty$ . Precisely,

$$\lim_{n \rightarrow \infty} z^n S_0 = \lim_{n \rightarrow \infty} S_n = z^\infty$$

There is a similar collection which limits to  $z^{-\infty}$  obtained from the "other half" of  $X$  mentioned above. The action of  $y$  on  $\partial X$  gives the following:

$$\lim_{n \rightarrow \infty} y^n z x^\infty = \lim_{n \rightarrow \infty} y^n z x^{-\infty} = y^\infty$$

This is true since the planes  $y^n P_1$  all share a common direction.

Observe that the circles  $\partial P_0$  and  $\partial P_n$  for  $n \geq 2$  satisfy the hypotheses of Lemma 6.1. The lemma implies that any geodesic line with endpoints in  $\partial P_n$  and  $\partial P_0$  must pass within a constant of the segment between the origins of these planes (since this segment realizes the distance between the planes).  $\square$

Now suppose  $a$  is a hyperbolic isometry of  $X$  and  $z \in \partial X$  with  $Td(z, \text{Fix}(\bar{a})) > \pi$ . There is a geodesic line  $\sigma = \sigma_{z a^{-\infty}}$  and neighborhoods  $U_{a^{-\infty}}$  and  $V_z$  as in Lemma 5.1. The following lemma shows that  $z$  and the points close to  $z$  in the cone topology converge to  $a^\infty$  under the action of  $a$  and that convergence is uniform on the closure of a cone topology neighborhood around  $z$ .

**Lemma 6.3.** *Suppose  $z \in \partial X$  with  $Td(z, \{a^{\pm\infty}\}) > \pi$ . If  $V_z$  and  $U_{a^{-\infty}}$  are chosen as in lemma 3.5, then for every (cone topology) neighborhood  $U \subset U_{a^{-\infty}}$  of  $a^\infty$ , there exists an  $N > 0$  such that for all  $n \geq N$ ,  $\bar{a}(\bar{V}_z) \subset U$ .*

*Proof.* Let  $U = U(a^\infty, R, \epsilon, a(0)) \subset U_{a^{-\infty}}$  be a basic open neighborhood around  $a^\infty$ . We know we can choose an  $N > 0$  so that for all  $n \geq N$ ,  $\bar{a} \cdot z \in U$  because  $z$  converges to  $a^\infty$  under the action of  $a$  (see Remark following Theorem 4.1).

Suppose the conclusion of the lemma does not hold. Then for each  $n > N$ , there exists a  $v_n \in V_z$  such that  $\bar{a} \cdot v_n$  is not in  $U$ . Without loss of generality, suppose  $v_n$  converges to  $v \in \bar{V}_z$ . Then for all  $n > 0$   $\bar{a}^n \cdot v$  is not in  $U$  by construction. We make the following observation:

*Observation:* Fix  $i > 0$  large enough so that  $a^{-i} \cdot a(0) \in U_{a^{-\infty}}$ , and consider the rays  $r_n$  from  $a^{-i} \cdot a(0)$  to  $v_n$ . These rays must converge to a ray  $r$  from  $a^{-i} \cdot a(0)$  to  $v$  and this ray passes within  $K$  (the  $K$  from Lemma 5.1) of  $\sigma(0)$ . Indeed, let  $m_n$  denote the point on  $v_n$  which passes within  $K$  of  $\sigma(0)$ . The  $m_n$  must converge to a point  $m$  in the closed  $K$  ball around  $\sigma(0)$  and  $m$  lies on the geodesic ray  $r$ .

Let  $D = d(a(0), \sigma(0))$ . Then we know  $d(a(0), m) \leq D + \overline{K}$  from the above observation. Also notice  $a^i \cdot r$  is the geodesic ray from  $a(0)$  to  $\overline{a^i} \cdot v$ . Consider the geodesic triangle with vertices  $a(0)$ ,  $a^i \cdot a(0)$ , and  $a^i \cdot m$ . The side  $[a^i \cdot a(0), a^i \cdot m]$  has length no more than  $D + K$  (independent of  $i$ ) because  $a$  is an isometry. The side  $[a(0), a^i \cdot a(0)]$  has length  $i|a|$ . Let  $s$  denote the remaining side. In a comparison triangle we can use similar triangles to obtain the following in  $X$ :

$$\frac{d(a(r), s(r))}{d(a(0), a(r))} = \frac{d(a^i \cdot a(0), a^i \cdot m)}{d(a(0), a^i \cdot a(0))} \leq \frac{D + K}{i|a|}$$

The last inequality can be chosen less than  $\epsilon$  by choosing  $i$  large enough, therefore  $\overline{a^i} \cdot v \in U$  for large  $i$ . This contradicts our choice of  $v$  and so the lemma holds.  $\square$

**Main Theorem.** *If  $a, b \in \Gamma$  are infinite order elements with  $Td(\{a^{\pm\infty}\}, \{a^{\pm\infty}\}) > \pi$ , then the subgroup generated by  $a$  and  $b$  contains a free subgroup. In fact, there exists an  $N > 0$  such that for all  $n \geq N$ ,  $a^n b^{-n}$  is rank one.*

*Proof.* Let  $\phi_n = a^n b^{-n}$ . In order to show the last claim in the statement of the theorem, it suffices to show  $\phi_n(x_0) \rightarrow a^\infty$  and  $\phi_n(x_0) \rightarrow b^\infty$  for one  $x_0 \in X$  by Lemma 5.3. Indeed, this will show that for large  $n$ ,  $\phi_n$  has an axis which does not bound a flat half plane - i.e.  $\phi_n$  is rank 1. Then we can apply Theorem 5.4 to get a free subgroup in the subgroup generated by  $a$  and  $b$ .

Let  $x_0 \in X$  be arbitrary and let  $W$  be an open set about  $a^\infty$ . By assumption, we can choose neighborhoods  $U$  of  $a^\infty$  and  $V$  or  $b^{-\infty}$  as in Lemma 5.1. Without loss of generality, assume  $W \subset U$ .

Choose  $N_1 > 0$  so that for all  $n \geq N_1$ ,  $b^{-n} \cdot x_0 \in V$ . By assumption,  $b^{-\infty}$  satisfies the hypotheses of Lemma 6.3. Thus choose  $N_2$  so that for all  $n \geq N_2$ ,  $\overline{a^n} \cdot \overline{V} \in W$ .

Let  $N$  be the larger of  $N_1, N_2$ . We have shown that for all  $n \geq N$ ,  $\phi_n(x_0) = a^n \cdot (b^{-n} \cdot x_0) \in W$  for any open set  $W$  about  $a^\infty$  as needed. A completely symmetric argument shows  $\phi_n^{-1}(x_0) \rightarrow b^\infty$ .  $\square$

## 7. SOME REMARKS ON THE COMMUTATIVE CASE

The following fact from [11] allows us to push through and if and only if statement in the case that  $X$  has rank 2 - i.e. there are no flats of dimension 3 or higher embedded isometrically into  $X$ . This theorem was known in the classical setting of nonpositively curved manifolds and can be found in [4]. Lemma 7.2 can also be found in [11] as Sublemma 2.3.

**Theorem 7.1.** *Suppose  $X$  is a proper CAT(0) space and  $S \subset \partial_{Tits} X$  a unit sphere which does not bound a unit hemisphere in  $\partial_{Tits} X$ . Then there exists a flat  $F \subset X$  with  $\partial F = S$ .*

**Lemma 7.2.** *Suppose  $F$  is a flat and  $Y$  is a closed, convex subset of  $X$  with  $\partial F \subset \partial Y$ . Then  $Y$  contains a flat which is parallel to  $F$ .*

*Proof.* If  $F \cap Y \neq \emptyset$ , then  $F \subset Y$  by convexity. Suppose  $y \in Y$  with  $d(F, Y) = d(y, F)$ . Consider the subset of  $Y$  consisting of all rays beginning at  $y$  and ending at the different points of  $F$ . This set is a flat contained in  $Y$  which is parallel to  $F$ . Indeed, let  $x \in F$  be the closest point projection of  $y$  onto  $F$  and let  $z \in \partial F$ . Then the ray  $r$  from  $y$  to  $z$  and the ray  $r'$  from  $x$  to  $z$  are asymptotic and cannot travel more than  $d(x, y)$  units apart at any time  $t > 0$  by convexity. But  $d(r(t), r'(t)) \geq d(x, y)$  for all  $t$  by choice of  $x$  and  $y$ , thus we have inequality and the lemma follows.  $\square$

For a hyperbolic isometry  $\gamma$ , denote by  $S_\gamma^0 = \{\gamma^{\pm\infty}\}$  the two fixed points of  $\bar{\gamma}$  which are the suspension points of  $\partial \text{Min}(\gamma)$ . Note that an isometric embedding of an  $S^1$  which contains  $S_a^0$  as an antipodal pair means that  $a^{\pm\infty} \subset S^1$  and  $Td(a^\infty, a^{-\infty}) = \pi$ .

**Theorem 7.3.** *Suppose  $\Gamma$  is a group acting geometrically on the rank 2 CAT(0) space  $X$  and suppose  $a, b \in \Gamma$  are infinite order elements. Then  $[a^p, b^q] = 1$  for  $p, q > 0$  if and only if there exists an isometric embedding of  $S^1$  into  $(\partial X, Td)$  with image in  $\text{Fix}(\bar{a}) \cap \text{Fix}(\bar{b})$  containing  $S_a^0$  and  $S_b^0$  as antipodal pairs.*

*Proof.* Suppose  $[a^p, b^q] = 1$  for  $p, q > 0$ . Then by the Flat Torus Theorem, there exists a plane  $P$  in  $\text{Min}(-a^p) \cap \text{Min}(b^q)$  on which the  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $a^p$  and  $b^q$



17. E. Swenson, *Subgroups of negatively curved groups*, preprint (1993).