

On the automorphisms of a graph product of abelian groups

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Abstract

We study the automorphisms of a graph product of finitely-generated abelian groups W . More precisely, we study a natural subgroup $\text{Aut}^* W$ of $\text{Aut} W$, with $\text{Aut}^* W = \text{Aut} W$ whenever vertex groups are finite and in a number of other cases. We prove a number of structure results, including a semi-direct product decomposition $\text{Aut}^* W = (\text{Inn} W \rtimes \text{Out}^0 W) \rtimes \text{Aut}^1 W$. We also give a number of applications, some of which are geometric in nature.

1 Introduction

The graph product of groups construction was first defined by Green [9]. It interpolates between the free product construction, in the case that Γ is a discrete graph, and the direct product construction, in the case that Γ is a complete graph. The class of graph products of finitely-generated abelian groups contains a number of important subclasses that are often treated separately. In the present article we pursue a unified treatment of the automorphisms of such groups. Our methods are combinatorial. Our results have a number of applications which are geometric in nature.

A non-trivial finite simplicial graph $\Gamma = \Gamma(V, E)$ is a pair consisting of a non-empty finite set $V = \{v_1, v_2, \dots, v_N\}$ (the *vertices*) and a set E (the *edges*) of unordered pairs from V . We say that vertices v_i, v_j are *adjacent* if $\{v_i, v_j\} \in E$. We consider Γ to be a metric object in the usual way, with d_Γ denoting the distance function. An *order map* (on Γ) is a function

$$\mathbf{m}: \{1, 2, \dots, N\} \rightarrow \{p^\alpha \mid p \text{ prime and } \alpha \in \mathbb{N}\} \cup \{\infty\}.$$

A pair (Γ, \mathbf{m}) is called a *labeled graph* and determines a group $W(\Gamma, \mathbf{m})$ with the following presentation (by convention, the relation v_i^∞ is the trivial relation):

$$\langle V \mid v_i^{\mathbf{m}(i)}, v_j v_k v_j^{-1} v_k^{-1} \ (1 \leq i, j, k \leq N, j < k, d_\Gamma(v_j, v_k) = 1) \rangle. \quad (1)$$

We say that $W(\Gamma, \mathbf{m})$ is a *graph product of directly-indecomposable cyclic groups*. Following an established convention, we do not distinguish between a vertex of Γ and the corresponding generator of $W(\Gamma, \mathbf{m})$.

The class of graph products of directly-indecomposable cyclic groups is identical to the class of graph products of finitely-generated abelian groups for the following reason: if G is group and G is isomorphic to a graph product of finitely-generated abelian groups, then there exists a unique isomorphism class of labeled-graphs (Γ, \mathbf{m}) such that $G \cong W(\Gamma, \mathbf{m})$ [12]. Empowered by this fact, we usually omit mention of Γ and \mathbf{m} from the notation, writing $W := W(\Gamma, \mathbf{m})$. The important subclasses alluded to in the opening paragraph include finitely-generated abelian groups (Γ a complete graph), graph products of primary cyclic groups ($\mathbf{m}(i) < \infty$ for each i), right-angled Coxeter groups ($\mathbf{m}(i) = 2$ for each i) and right-angled Artin groups ($\mathbf{m}(i) = \infty$ for each i).

For a full subgraph Δ of Γ , we write $W(\Delta)$ for the subgroup (known as a *special subgroup*) of W generated by the vertices in Δ . We write $\text{MCS}(\Gamma)$ for the set of maximal complete subgraphs (or cliques) of Γ . The subgroups $W(\Delta)$, $\Delta \in \text{MCS}(\Gamma)$, will be called the *maximal complete subgroups* of W . If W is a graph product of primary cyclic groups, then the maximal complete subgroups are a set of representatives for the conjugacy classes of maximal finite subgroups of W [9, Lemma 4.5] and each automorphism of W maps each maximal complete subgroup to a conjugate of some maximal complete subgroup. This is not true in an arbitrary graph product of directly-indecomposable cyclic groups, but we may pretend that it is by restricting our attention to a natural subgroup of $\text{Aut } W$.

Definition 1.1. Write $\text{Aut}^* W$ for the subgroup of $\text{Aut} W$ consisting of those automorphisms which map each maximal complete subgroup to a conjugate of a maximal complete subgroup.

The following lemma is immediate from the discussion above and the main result of [18]. For each $1 \leq i \leq N$, we write L_i (resp. S_i) for the *link* (resp. *star*) of v_i .

Lemma 1.2. *If W is a graph product of directly-indecomposable cyclic groups, then $\text{Aut}^* W = \text{Aut} W$ in each of the following cases:*

1. W is a graph product of primary cyclic groups;
2. W is a right-angled Artin group and $L_i \not\subseteq L_j$ for each pair of distinct non-adjacent vertices $v_i, v_j \in V$;
3. W is a right-angled Artin group and Γ contains no vertices of valence less than two and no circuits of length less than 5.

Remark 1.3. Case (2) can be substantially generalized to groups that are not right-angled Artin groups.

We now report the main results of the present article. They concern the structure of $\text{Aut}^* W$ and shall make reference to the subgroups and quotients of $\text{Aut} W$ defined in Figure 1. In writing $\text{Aut}^0 W$ for the subgroup of ‘conjugating automorphisms’, we follow Tits [26]. Mühlherr [24] writes $\text{Spe}(W)$ for the same subgroup. Charney, Crisp and Vogtmann [5] use the notation $\text{Aut}^0 W$ and $\text{Out}^0 W$ for different subgroups of the automorphism group of a right-angled Artin group than described here.

Tits [26] proved that if W is a right-angled Coxeter group, then $\text{Aut} W = \text{Aut}^0 W \rtimes \text{Aut}^1 W$. Our first main result is a generalization of Tits’ splitting.

Theorem 1.4 (cf. [26]). *If W is a graph product of directly-indecomposable cyclic groups, then*

$$\text{Aut}^* W = \text{Aut}^0 W \rtimes \text{Aut}^1 W.$$

If W_{ab} denotes the abelianization of W , then the subgroup $\text{Aut}^1 W$ is isomorphic to the image of $\text{Aut}^* W$ under the natural map $\text{Aut} W \rightarrow \text{Aut} W_{\text{ab}}$. In particular, $\text{Aut}^1 W$ is finite in the case that W is a graph product of primary cyclic groups.

Group	Description
$\text{Aut}^* W$	Those automorphisms of W which map each maximal complete subgroup to a conjugate of a maximal complete subgroup
$\text{Aut}^1 W$	Those automorphisms of W which map each maximal complete subgroup to a maximal complete subgroup
$\text{Aut}^0 W$	Those automorphisms of W which map each vertex $v_i \in V$ to a conjugate of itself
$\text{Inn } W$	The inner automorphisms of W
$\text{Out}^0 W$	The subgroup of $\text{Aut } W$ generated by the set \mathcal{P}^0
$\mathcal{O}ut W$	The quotient $\text{Aut } W / \text{Inn } W$
$\mathcal{O}ut^0 W$	The quotient $\text{Aut}^0 W / \text{Inn } W$

Figure 1: Subgroups and quotients of $\text{Aut } W$.

We next turn our attention to the study of $\text{Aut}^0 W$. Choosing the right generating set for various subgroups of $\text{Aut}^0 W$ shall be important throughout. For $1 \leq i \leq N$ and K a (non-trivial) connected component of $\Gamma \setminus S_i$, we write χ_{iK} for the automorphism of W determined by

$$\chi_{iK}(v_j) = \begin{cases} v_i v_j v_i^{-1} & \text{if } v_j \in K, \\ v_j & \text{if } v_j \notin K. \end{cases}$$

Such an automorphism is called a *partial conjugation* with *operating letter* v_i and *domain* K . We write \mathcal{P} for the set of partial conjugations (see §A for an example). Laurence [17, Theorem 4.1] proved that $\text{Aut}^0 W$ is generated by \mathcal{P} .

For a subgraph $\Omega \subseteq \Gamma$, we write pr_Ω for the retraction map $W \rightarrow W(\Omega)$ and $\mathcal{P}_\Omega := \{\chi_{iQ} \in \mathcal{P} \mid v_i \in \Omega\}$. For $\phi \in \text{Aut}^0 W$ and $w_1, \dots, w_N \in W$ such that $\phi(v_i) = w_i v_i w_i^{-1}$ for each $1 \leq i \leq N$, we write ϕ_Ω for the map $V \rightarrow W$ defined by

$$v_i \mapsto pr_\Omega(w_i) \cdot v_i \cdot pr_\Omega(w_i)^{-1} \text{ for each } 1 \leq i \leq N.$$

We shall show that ϕ_Ω extends to an automorphism of W , also denoted by ϕ_Ω . In fact, the following holds:

Theorem 1.5. *For each subgraph $\Omega \subseteq \Gamma$, the map $\phi \mapsto \phi_\Omega$ is a retraction homomorphism $\text{Aut}^0 W \rightarrow \langle \mathcal{P}_\Omega \rangle$.*

The following immediate corollary is key to a number of our arguments.

Corollary 1.6 (The Restricted Alphabet Rewriting Lemma). *If $\phi \in \text{Aut}^0 W$ and there exist $z_1, \dots, z_N \in W(\Omega)$ such that $\phi(v_j) = z_j v_j z_j^{-1}$ for each $1 \leq j \leq N$, then any word for ϕ in the alphabet $\mathcal{P}^{\pm 1}$ may be rewritten as a word in the alphabet $\mathcal{P}_\Omega^{\pm 1}$ (a word which still spells ϕ) by simply omitting those generators not in $\mathcal{P}_\Omega^{\pm 1}$.*

Next we define a subset $\mathcal{P}^0 \subset \mathcal{P}$ (see Definition 4.6) such that $\text{Aut}^0 W$ is generated by the disjoint union of \mathcal{P}^0 and the inner automorphisms. Using The Restricted Alphabet Rewriting Lemma we show that $\text{Inn } W \cap \langle \mathcal{P}^0 \rangle = \{id\}$ and hence prove our next main result.

Theorem 1.7. *If W is a graph product of directly-indecomposable cyclic groups, then*

$$\text{Aut}^0 W = \text{Inn } W \rtimes \text{Out}^0 W.$$

Consequently, $\text{Out}^0 W \cong \text{Out}^0 W$.

The subgroup $\text{Inn } W$ is well-understood and is itself a special subgroup of W (see Lemma 2.5)

In §8 we give sufficient conditions for the splittings of Theorems 1.4 and 1.7 to be compatible (Lemma 8.8) and then record an application to the theory of group extensions (Corollary 8.9 and Corollary 8.10).

To each partial conjugation we associate a subset of V called the set of *link points* of the partial conjugation (see Definition 5.1). If Γ is connected, then elements of \mathcal{P}^0 which do not share a common link point must commute (see Lemma 6.2 and Remark 6.3). So, in anticipation of studying $\text{Out}^0 W$, it is natural to consider sets of partial conjugations which share a common link point. We prove the following:

Theorem 1.8. *If W is a graph product of directly-indecomposable cyclic groups and $1 \leq i \leq N$ and $\mathcal{L}_i \subset \mathcal{P}$ is the set of partial conjugations for which v_i is a link point, then the natural restriction homomorphism $\rho_i : \langle \mathcal{L}_i \rangle \hookrightarrow \text{Aut}^0 W(L_i)$ is injective.*

We now consider $\text{Out}^0 W$. We give a simple graph criterion which characterizes when $\text{Out}^0 W$ is abelian.

Definition 1.9 (SIL). We say that Γ contains a *separating intersection of links* (SIL) if there exist $1 \leq i < j \leq N$ such that the following conditions hold:

1. $d(v_i, v_j) \geq 2$;
2. there exists a connected component R of $\Gamma \setminus (L_i \cap L_j)$ such that $v_i, v_j \notin R$.

Theorem 1.10. *If W is a graph product of directly-indecomposable cyclic groups, then the following are equivalent:*

1. $\text{Out}^0 W$ is an abelian group;
2. Γ does not contain a SIL.

In the case that W is a graph product of primary cyclic groups, this also characterizes when $\mathcal{O}ut W$ is finite.

Corollary 1.11. *If W is a graph product of primary cyclic groups, then the following are equivalent:*

1. $\text{Out}^0 W$ is an abelian group;
2. Γ does not contain a SIL;
3. $\mathcal{O}ut W$ is finite.

In §8 we describe a number of applications of Corollary 1.11 to the study of the geometry of W and $\text{Aut} W$. We examine the combined effect of the existence (or absence) of an SIL and certain other graph properties which are known to determine geometric properties of W such as word-hyperbolicity (Corollary 8.1), the isolated flats property (Corollary 8.4) and whether or not W can act on a CAT(0) space with locally-connected visual boundary (Corollary 8.5). We also characterize when $\text{Aut} W$ is word hyperbolic (Corollary 8.7).

In the special case that Γ is a tree, our study of $\text{Out}^0 W$ can proceed much further. In this case, each partial conjugation has a unique link point, the subsets $\mathcal{L}_i^0 := \mathcal{L}_i \cap \mathcal{P}^0$ partition \mathcal{P}^0 and the partition corresponds to a direct product decomposition of $\text{Out}^0 W$. Since each $W(L_i)$ is a free product of cyclic groups and each image $\rho_i(\langle \mathcal{L}_i^0 \rangle)$ is easily understood (see Proposition 7.3), we are able to give a complete description of $\text{Out}^0 W$.

Theorem 1.12. *If W is a graph product of directly-indecomposable cyclic groups and Γ is a tree, then*

$$\text{Out}^0 W \cong \text{Ab} \times \left(\prod_{i=1}^N \text{Out}^0 W(L_i) \right)$$

for a finitely-generated abelian group Ab as described in Remark 7.5. In particular:

1. if W is a right-angled Artin group, then Ab is a free abelian group;
2. if W is a graph product of primary cyclic groups, then Ab is a finite abelian group.

We apply Theorem 1.12 to determine a finite presentation for $\text{Aut}^* W$ (Remark 7.5), to calculate the virtual cohomological dimension of $\mathcal{O}ut W$ in the case that W is a graph product of primary cyclic groups (Corollary 8.12) and to prove the existence of regular languages of normal forms for $\text{Out}^0 W$ and $\text{Aut}^0 W$ in the case that W is a right-angled Artin group (Corollary 8.13).

We mention that Laurence [17] and Mühlherr [24] independently determined finite presentations for $\text{Aut}^0 W$ in the case that W is a right-angled Coxeter group. Castella [4] determined a finite presentation of $\text{Aut}^1 W$ for a certain subclass of right-angled Coxeter groups. Proving a conjecture of Servatius [25], Laurence [18] determined a generating set for $\text{Aut} W$ in the case that W is a right-angled Artin group—a generating set for $\text{Aut}^* W$ can be deduced from this list. To the best of the authors' knowledge, Laurence's unpublished Ph.D. Thesis [17] is the only previous work to consider the automorphisms of graph products of abelian groups in a unified way.

We now briefly describe the structure of the present article. We attend to some preliminaries in §2. Theorem 1.4 is the topic of §3. Theorem 1.5 is proved in §4.1. The Restricted Alphabet Rewriting Lemma is then used to prove Theorem 1.7 in §4.2. Theorem 1.8 is proved in §5. Theorem 1.10 and Corollary 1.11 are proved in §6. Theorem 1.12 is proved in §7. We describe a number of applications of our results in §8. In §A we illustrate a number of definitions and results by following an example. Each section is prefaced by a short description of its contents.

2 Preliminaries

In this section we establish notation and remind the reader of some fundamental results concerning graph products of groups. We have stated the results only for the class of graph products of directly-indecomposable cyclic groups. They appear, in more general form, in Elisabeth Green's Ph.D. Thesis [9] and Michael Laurence's Ph.D. Thesis [17], proved by different methods.

By a subgraph of Γ we shall always mean a full subgraph. Thus a subgraph $\Delta = (V_\Delta, E_\Delta)$ is determined by a subset $V_\Delta \subseteq V$ and the rule $E_\Delta = \{\{v_i, v_j\} \in E \mid v_i, v_j \in V_\Delta\}$.

We remind the reader that, for each $1 \leq i \leq N$, we write L_i (resp. S_i) for the *link* (resp. *star*) of v_i . That is, L_i (resp. S_i) is the subgraph of Γ generated by the vertices $\{v_j \in V \mid d(v_i, v_j) = 1\}$ (resp. $\{v_j \in V \mid d(v_i, v_j) \leq 1\}$).

Definition 2.1. Let $1 \leq i_1, \dots, i_k \leq N$ be such that $i_j \neq i_{j+1}$ and let $\alpha_1, \dots, \alpha_k$ be non-zero integers. Each $v_{i_j}^{\alpha_j}$ is a *syllable* of the word $v_{i_1}^{\alpha_1} v_{i_2}^{\alpha_2} \dots v_{i_k}^{\alpha_k}$, and we say that the word is *reduced* if there is no word with fewer syllables which spells the same element of W . We say that consecutive syllables $v_{i_j}^{\alpha_j}$, $v_{i_{j+1}}^{\alpha_{j+1}}$ are *adjacent* if v_{i_j} and $v_{i_{j+1}}$ are.

Lemma 2.2 (The Deletion Condition). *Let $1 \leq i_1, \dots, i_k \leq N$ be such that $i_j \neq i_{j+1}$ and let $\alpha_1, \dots, \alpha_k$ be non-zero integers. If the word $v_{i_1}^{\alpha_1} v_{i_2}^{\alpha_2} \dots v_{i_k}^{\alpha_k}$ is not reduced, then there exist p, q such that $1 \leq p < q \leq k$, $v_{i_p} = v_{i_q}$ and v_{i_p} is adjacent to each vertex $v_{i_{p+1}}, v_{i_{p+2}}, \dots, v_{i_{q-1}}$.*

Lemma 2.3 (Normal Form). *Let $1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq N$ and $\alpha_1, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k \in \mathbb{Z} \setminus \{0\}$. If $v_{i_1}^{\alpha_1} v_{i_2}^{\alpha_2} \dots v_{i_k}^{\alpha_k}$ and $w_{j_1}^{\beta_1} w_{j_2}^{\beta_2} \dots w_{j_k}^{\beta_k}$ are reduced words which spell the same element of W , then the first word may be transformed into the second by repeatedly swapping the order of adjacent syllables.*

The following three results witness the importance of special subgroups to the study of W .

Lemma 2.4. *Let Δ be a subgraph of Γ . The natural map $W(\Delta) \rightarrow W(\Gamma)$ is an embedding.*

Lemma 2.5. *The center of W is the special subgroup generated by the vertices $\{v_i \in V \mid d_\Gamma(v_i, v_j) \leq 1 \text{ for each } 1 \leq j \leq N\}$. Further, the center of W is finite in the case that W is a graph product of primary cyclic groups.*

It follows that $\text{Inn } W$ is isomorphic to the special subgroup generated by the vertices $\{v_i \in V \mid d_\Gamma(v_i, v_j) > 1 \text{ for some } 1 \leq j \leq N\}$. Further, $\text{Inn } W$ is isomorphic to a finite-index subgroup of W in the case that W is a graph product of primary cyclic groups.

Lemma 2.6. *A special subgroup $W(\Delta)$ has finite order if and only if Δ is a complete graph and each vertex of Δ has finite order. Further, if a subgroup H of W has finite order, then H is contained in some conjugate of a special subgroup of finite order.*

The centralizer of a vertex is easily understood.

Lemma 2.7. *For each $1 \leq j \leq N$, the centralizer of v_j in W is the special subgroup generated by S_j .*

3 A splitting of $\text{Aut}^* W$

In this section we prove that $\text{Aut}^* W = \text{Aut}^0 W \rtimes \text{Aut}^1 W$ (Theorem 1.4), thus generalizing a result of Tits [26]. We shall prove the result by exhibiting a retraction homomorphism $\text{Aut}^* W \rightarrow \text{Aut}^1 W$ with kernel $\text{Aut}^0 W$.

For each $\Delta \in \text{MCS}(\Gamma)$ and $w \in W$ and $u \in wW(\Delta)w^{-1}$, there exists a unique element $[u]$ of minimal length in the conjugacy class of u . Equivalently, $[u]$ is the unique element of $W(\Delta)$ in the conjugacy class of u . For each automorphism $\gamma \in \text{Aut}^* W$, define $r(\gamma) : V \rightarrow W$ by the rule $v \mapsto [\gamma(v)]$.

Lemma 3.1. *For each automorphism $\gamma \in \text{Aut}^* W$, the map $r(\gamma) : V \rightarrow W$ extends to an endomorphism of W .*

Proof. It suffices to show that the relations used to define W are ‘preserved’ by $r(\gamma)$.

Let $1 \leq i \leq N$ be such that $\mathbf{m}(i) < \infty$. Since the order of an element is preserved under automorphisms and conjugation, $(r(\gamma)(v_i))^{\mathbf{m}(v_i)} = 1$ and $r(\gamma)$ preserves the relation $v_i^{\mathbf{m}(v_i)} = 1$.

Let $1 \leq j < k \leq N$ be such that $d(v_j, v_k) = 1$. There exist $w \in W$ and $\Delta \in \text{MCS}(\Gamma)$ and $a, b \in W(\Delta)$ such that $\gamma(v_j) = waw^{-1}$ and $\gamma(v_k) = bw^{-1}$. Recall that $W(\Delta)$ is an abelian group. Then

$$(r(\gamma)(v_j))(r(\gamma)(v_k))(r(\gamma)(v_j))^{-1}(r(\gamma)(v_k))^{-1} = aba^{-1}b^{-1} = 1.$$

Thus $r(\gamma)$ preserves the relation $v_j v_k v_j^{-1} v_k^{-1} = 1$. □

We abuse notation by writing $r(\gamma) : W \rightarrow W$ for the endomorphism of W determined by $r(\gamma) : V \rightarrow W$.

Lemma 3.2. *For each automorphism $\delta \in \text{Aut}^* W$ and $\Delta \in \text{MCS}(\Gamma)$ and $a \in W(\Delta)$, we have $r(\delta)(a) = [\delta(a)]$.*

Proof. We have $a = d_1^{\epsilon_1} \dots d_q^{\epsilon_q}$ for some vertices d_1, \dots, d_q in Δ and some integers $\epsilon_1, \dots, \epsilon_q$. By the definition of $\text{Aut}^* W$, there exist $w \in W$ and $\Theta \in \text{MCS}(\Gamma)$ such that $\delta(W(\Delta)) = wW(\Theta)w^{-1}$. Hence there exist $t_1, \dots, t_q \in W(\Theta)$ such that $\delta(d_i) = wt_i w^{-1}$ for each $1 \leq i \leq q$. Then

$$r(\delta)(a) = r(\delta)(d_1)^{\epsilon_1} \dots r(\delta)(d_q)^{\epsilon_q} = t_1^{\epsilon_1} \dots t_q^{\epsilon_q} = [wt_1^{\epsilon_1} w^{-1} \dots wt_q^{\epsilon_q} w^{-1}] = [\delta(a)],$$

as required. □

Lemma 3.3. *For each pair of automorphisms $\gamma, \delta \in \text{Aut}^* W$, we have $r(\delta\gamma) = r(\delta)r(\gamma)$.*

Proof. Let $\gamma, \delta \in \text{Aut}^* W$ and let $1 \leq i \leq N$. There exist $\Delta \in \text{MCS}(\Gamma)$ and $a \in W(\Delta)$ and $w_1 \in W$ such that $\gamma(v_i) = w_1 a w_1^{-1}$. There exist $\Theta \in \text{MCS}(\Gamma)$, $b \in W(\Theta)$ and $w_2 \in W$ such that $\delta(W(\Delta)) = w_2 W(\Theta) w_2^{-1}$ and $\delta(a) = w_2 b w_2^{-1}$. By Lemma 3.2 we have that $r(\delta)(a) = b$. Then

$$r(\delta\gamma)(v_i) = [\delta\gamma(v_i)] = [\delta(w_1)w_2 b w_2^{-1} \delta(w_1)^{-1}] = b = r(\delta)(a) = r(\delta)r(\gamma)(v_i),$$

as required. \square

Lemma 3.4. *For each $\gamma \in \text{Aut}^* W$, $r(\gamma) \in \text{Aut}^1 W$.*

Proof. Let $\gamma \in \text{Aut}^* W$. By Lemma 3.3, $r(\gamma^{-1})r(\gamma) = r(\gamma^{-1} \circ \gamma) = r(\text{id}) = \text{id}$ and $r(\gamma)$ is an automorphism of W . It is clear from the definitions that $r(\gamma) \in \text{Aut}^1 W$. \square

Proposition 3.5. *The map r is a retraction homomorphism $\text{Aut}^* W \rightarrow \text{Aut}^1 W$ with kernel $\text{Aut}^0 W$.*

Proof. By Lemmas 3.3 and 3.4, r is a homomorphism $\text{Aut}^* W \rightarrow \text{Aut}^1 W$. It is clear from the definitions that r restricts to the identity map on $\text{Aut}^1 W$ and r has kernel $\text{Aut}^0 W$. \square

Remark 3.6 (Tits' approach and Theorem 1.4). For each $\Delta \in \text{MCS}(\Gamma)$, we may consider $W(\Delta)$ as a subgroup of W_{ab} . Then the union

$$G = \bigcup_{\Delta \in \text{MCS}(\Gamma)} W(\Delta) \subset W_{\text{ab}}$$

is a groupoid in the usual way. In case W is a right-angled Coxeter group, Tits [26] identifies $\text{Aut}^1 W$ with the groupoid automorphisms $\text{Aut} G$ of G and constructs a section of the obvious homomorphism $\text{Aut}^* W \rightarrow \text{Aut} G$. This identification carries over in the case that W is an arbitrary graph product of directly-indecomposable groups and a section of the homomorphism $\text{Aut}^* W \rightarrow \text{Aut} G$ is defined similarly.

4 The group $\text{Aut}^0 W$

The primary goal of this section is to prove that $\text{Aut}^0 W = \text{Inn} W \rtimes \text{Out}^0 W$ (Theorem 1.7). Before achieving this in §4.2, we prove The Restricted Alphabet Rewriting Lemma in §4.1.

We remind the reader that $\text{Aut}^0 W$ is generated by the set of partial conjugations \mathcal{P} .

Remark 4.1. Although we shall not assume that Γ is connected throughout, it is occasionally convenient to note that such an assumption places no restriction on our study of $\text{Aut}^0 W$. For suppose that Γ is not connected. Write (Γ^+, \mathbf{m}^+) for the labeled-graph obtained from (Γ, \mathbf{m}) as follows:

1. introduce a new vertex v_0 and extend \mathbf{m} to a function \mathbf{m}^+ with domain $\{0, 1, 2, \dots, N\}$ by making a choice

$$\mathbf{m}(0) \in \{p^\alpha \mid p \text{ prime and } \alpha \in \mathbb{N}\} \cup \{\infty\};$$

2. add an edge from v_0 to v_i for each $1 \leq i \leq N$.

Write $W^+ := W(\Gamma^+, \mathbf{m}^+)$ and write S_i^+ for the star of v_i in Γ^+ . For each $1 \leq i \leq N$, the connected components of $\Gamma \setminus S_i$ are identical to the connected components of $\Gamma^+ \setminus S_i^+$. The subgraph $\Gamma^+ \setminus S_0^+$ is empty. Since \mathcal{P} generates $\text{Aut}^0 W$, it follows that $\text{Aut}^0 W \cong \text{Aut}^0 W^+$.

4.1 The Restricted Alphabet Rewriting Lemma

In this subsection we prove Theorem 1.5. As an aside, we illustrate the power of The Restricted Alphabet Rewriting Lemma by proving that \mathcal{P} is a minimal generating set for $\text{Aut}^0 W$.

Throughout, we fix a subgraph $\Omega \subseteq \Gamma$. Recall the definitions of pr_Ω , \mathcal{P}_Ω and ϕ_Ω given in the Introduction.

Lemma 4.2. *The map $\phi \mapsto \phi_\Omega$ is well-defined*

Proof. Let $\phi \in \text{Aut}^0 W$ and $w_1, \dots, w_N \in W$ and $u_1, \dots, u_N \in W$ be such that $\phi(v_i) = w_i v_i w_i^{-1} = u_i v_i u_i^{-1}$ for each $1 \leq i \leq N$. Fix $1 \leq i \leq N$. We must show that $pr_\Omega(w_i) \cdot v_i \cdot pr_\Omega(w_i)^{-1} = pr_\Omega(u_i) \cdot v_i \cdot pr_\Omega(u_i)^{-1}$. Since $w_i v_i w_i^{-1} = u_i v_i u_i^{-1}$, we have that $w_i^{-1} u_i$ is in the centralizer of v_i . Recall that the centralizer of v_i is generated by S_i (Lemma 2.7). Thus there exists $z_i \in \langle S_i \rangle$ such that $u_i = w_i z_i$. Since $pr_\Omega(S_i) \subseteq S_i \cup \{id\}$, $pr_\Omega(z_i) \in \langle S_i \rangle$ and we have

$$\begin{aligned} pr_\Omega(w_i) \cdot v_i \cdot pr_\Omega(w_i)^{-1} &= pr_\Omega(u_i z_i) \cdot v_i \cdot pr_\Omega(u_i z_i)^{-1} \\ &= pr_\Omega(u_i) \cdot pr_\Omega(z_i) \cdot v_i \cdot pr_\Omega(z_i)^{-1} \cdot pr_\Omega(u_i)^{-1} \\ &= pr_\Omega(u_i) \cdot v_i \cdot pr_\Omega(u_i)^{-1}. \end{aligned}$$

□

Lemma 4.3. *For each $\phi \in \text{Aut}^0 W$, the map $\phi_\Omega : V \rightarrow W$ extends to a homomorphism $\phi_\Omega : W \rightarrow W$.*

Proof. Let $\phi \in \text{Aut}^0 W$ and $w_1, \dots, w_N \in W$ be such that $\phi(v_i) = w_i v_i w_i^{-1}$ for each $1 \leq i \leq N$. We must show that the map ϕ_Ω ‘preserves’ the defining relations of W .

Let $1 \leq i \leq N$. Since $\phi_\Omega(v_i)$ is conjugate to v_i , it has the same order as v_i and the relation $v_i^{\mathbf{m}(i)}$ is preserved.

Let $1 \leq i < j \leq N$ be such that v_i and v_j are adjacent. Since $\phi \in \text{Aut}^0 W$, it follows that there exists $w \in W$ such that $\phi(v_i) = w v_i w^{-1}$ and $\phi(v_j) = w v_j w^{-1}$. Then $w_i = w z_i$ for some $z_i \in \langle S_i \rangle$ and $w_j = w z_j$ for some $z_j \in \langle S_j \rangle$. So $w_i^{-1} w_j = z_i^{-1} z_j$ and $\text{pr}_\Omega(w_i^{-1}) \cdot \text{pr}_\Omega(w_j) = \text{pr}_\Omega(z_i^{-1}) \cdot \text{pr}_\Omega(z_j)$. We have

$$\begin{aligned}
& \phi_\Omega(v_i) \cdot \phi_\Omega(v_j) \cdot \phi_\Omega(v_i)^{-1} \cdot \phi_\Omega(v_i)^{-1} \\
= & \text{pr}_\Omega(w_i) \cdot v_i \cdot \text{pr}_\Omega(w_i^{-1}) \cdot \text{pr}_\Omega(w_j) \cdot v_j \cdot \text{pr}_\Omega(w_j)^{-1} \\
& \quad \cdot \text{pr}_\Omega(w_i) \cdot v_i^{-1} \cdot \text{pr}_\Omega(w_i)^{-1} \cdot \text{pr}_\Omega(w_j) \cdot v_j^{-1} \cdot \text{pr}_\Omega(w_j)^{-1} \\
= & \text{pr}_\Omega(w_i) \cdot v_i \cdot \text{pr}_\Omega(z_i^{-1}) \cdot \text{pr}_\Omega(z_j) \cdot v_j \cdot \text{pr}_\Omega(z_j)^{-1} \\
& \quad \cdot \text{pr}_\Omega(z_i) \cdot v_i^{-1} \cdot \text{pr}_\Omega(z_i)^{-1} \cdot \text{pr}_\Omega(z_j) \cdot v_j^{-1} \cdot \text{pr}_\Omega(w_j)^{-1} \\
= & \text{pr}_\Omega(w_i) \cdot \text{pr}_\Omega(z_i^{-1}) \cdot v_i v_j v_i^{-1} v_j^{-1} \cdot \text{pr}_\Omega(z_j) \cdot \text{pr}_\Omega(w_j)^{-1} \\
= & \text{pr}_\Omega(w_i z_i^{-1}) \cdot 1 \cdot \text{pr}_\Omega(w_j z_j^{-1})^{-1} \\
= & \text{pr}_\Omega(w) \cdot \text{pr}_\Omega(w)^{-1} \\
= & 1,
\end{aligned}$$

and the relation $v_i v_j v_i^{-1} v_j^{-1}$ is preserved. \square

Lemma 4.4. *For $\phi, \theta \in \text{Aut}^0 W$, $(\phi \circ \theta)_\Omega = \phi_\Omega \circ \theta_\Omega$.*

Proof. Fix $1 \leq i \leq N$. It suffices to show that $(\theta \circ \phi)_\Omega(v_i) = \theta_\Omega \circ \phi_\Omega(v_i)$. Write V^* for the set of (not necessarily reduced) words in the alphabet $V^{\pm 1}$. Let $\mathcal{W}_1, \dots, \mathcal{W}_N \in V^*$ and $\mathcal{U}_1, \dots, \mathcal{U}_N \in V^*$ be such that $\phi(v_j) = \mathcal{W}_j v_j \mathcal{W}_j^{-1}$ and $\theta(v_j) = \mathcal{U}_j v_j \mathcal{U}_j^{-1}$ for each $1 \leq j \leq N$. Let \mathcal{T}_i be the word constructed from \mathcal{W}_i as follows:

1. for each $1 \leq j \leq N$ and $\delta = \pm 1$, replace each occurrence of v_j^δ by $\mathcal{U}_j v_j^\delta \mathcal{U}_j^{-1}$;
2. append the word \mathcal{U}_i to the resulting word;

3. omit those letters not in Ω from the resulting word.

It is clear that $\mathcal{T}_i = pr_\Omega(\theta(\mathcal{W}_i)\mathcal{U}_i)$ and hence $(\theta\phi)_\Omega(v_i) = \mathcal{T}_i v_i \mathcal{T}_i^{-1}$ (with equality in W). Observe the following in the construction of \mathcal{T}_i :

(OB1) if $v_j \notin \Omega$, then each occurrence of v_j^δ in \mathcal{W}_j is eventually replaced by the word $pr_\Omega(U_j).pr_\Omega(U_j)^{-1}$, which is, of course, trivial in W ;

(OB2) if $v_j \in \Omega$, then each occurrence of v_j^δ in \mathcal{W}_j is eventually replaced by the word $pr_\Omega(U_j).v_j^\delta.pr_\Omega(U_j)^{-1}$.

Let \mathcal{T}'_i be the word constructed from \mathcal{W}_i as follows:

1. for each $1 \leq j \leq N$ such that $v_j \notin \Omega$ and each $\delta = \pm 1$, omit each occurrence of v_j^δ ;
2. for each $1 \leq j \leq N$ such that $v_j \in \Omega$ and each $\delta = \pm 1$, replace each occurrence of v_j^δ by $pr_\Omega(\mathcal{U}_j).v_j^\delta.pr_\Omega(\mathcal{U}_j)^{-1}$;
3. append the word $pr_\Omega(\mathcal{U}_i)$ to the resulting word.

It follows from (OB1) and (OB2) that $\mathcal{T}'_i = \mathcal{T}_i$ (with equality in W). It is clear from the construction of \mathcal{T}'_i that $\mathcal{T}'_i = (\theta_\Omega \circ pr_\Omega(\mathcal{W}_i)).pr_\Omega(\mathcal{U}_i)$ (with equality in W). We calculate the following (with all equalities in W):

$$\begin{aligned}
(\theta\phi)_\Omega(v_i) &= \mathcal{T}_i v_i \mathcal{T}_i^{-1} \\
&= \mathcal{T}'_i v_i (\mathcal{T}'_i)^{-1} \\
&= (\theta_\Omega \circ pr_\Omega(\mathcal{W}_i)).pr_\Omega(\mathcal{U}_i).v_i.pr_\Omega(\mathcal{U}_i)^{-1}.(\theta_\Omega \circ pr_\Omega(\mathcal{W}_i))^{-1} \\
&= \theta_\Omega(pr_\Omega(\mathcal{W}_i)).\theta_\Omega(v_i).\theta_\Omega(pr_\Omega(\mathcal{W}_i))^{-1} \\
&= \theta_\Omega(pr_\Omega(\mathcal{W}_i).v_i.pr_\Omega(\mathcal{W}_i)^{-1}) \\
&= \theta_\Omega \circ \phi_\Omega(v_i).
\end{aligned}$$

□

We now prove the main result of the subsection.

Proof of Theorem 1.5. By Lemma 4.4, $(\phi^{-1})_\Omega \circ \phi_\Omega = (\phi^{-1} \circ \phi)_\Omega = id_\Omega = id$ and ϕ_Ω is an automorphism of W . So the map $\phi \mapsto \phi_\Omega$ is a map $\text{Aut}^0 W \rightarrow \text{Aut}^0 W$. It follows from Lemma 4.4 that $\phi \mapsto \phi_\Omega$ is a homomorphism $\text{Aut}^0 W \rightarrow \text{Aut}^0 W$. It is clear from the definitions that

$$(\chi_{iK})_\Omega = \begin{cases} \chi_{iK} & \text{if } \chi_{iK} \in \mathcal{P}_\Omega \\ id & \text{if } \chi_{iK} \notin \mathcal{P}_\Omega. \end{cases}$$

It follows that $\phi \mapsto \phi_\Omega$ is a retraction homomorphism $\text{Aut}^0 W \rightarrow \langle \mathcal{P}_\Omega \rangle$. □

We conclude this subsection by noting the following immediate application of The Restricted Alphabet Rewriting Lemma.

Corollary 4.5. *The set \mathcal{P} is a minimal generating set for $\text{Aut}^0 W$.*

Proof. Let $\chi_{iK} \in \mathcal{P}$ and let \mathcal{U} be a word in the alphabet $\mathcal{P}^{\pm 1}$ such that $\mathcal{U} = \chi_{iK}$ (with equality in $\text{Aut}^0 W$). It follows from the Restricted Alphabet Rewriting Lemma that, simply by omitting some letters, \mathcal{U} may be rewritten as a word \mathcal{U}' in the alphabet

$$\{\chi_{iQ} \mid Q \text{ a connected component of } \Gamma \setminus S_i\}^{\pm 1}.$$

But the letters in \mathcal{U}' commute pairwise, so we must have that χ_{iK} appears with exponent sum 1 in \mathcal{U}' and hence also with exponent sum 1 in \mathcal{U} . Thus no word in the alphabet

$$\left(\mathcal{P} \setminus \{\chi_{iK}\}\right)^{\pm 1}$$

can spell χ_{iK} . □

4.2 A splitting of $\text{Aut}^0 W$

We now define a subset $\mathcal{P}^0 \subset \mathcal{P}$. We will show that $\text{Out}^0 W$, the subgroup generated by \mathcal{P}^0 , does not include any non-trivial inner automorphisms and that it is isomorphic to $\text{Out}^0 W$. Informally, one might understand the construction of \mathcal{P}^0 from \mathcal{P} as removing ‘just enough’ automorphisms to prevent the elements of $(\mathcal{P}^0)^{\pm 1}$ from spelling a non-trivial inner-automorphism.

Let \mathcal{I} denote the following set of inner automorphisms

$$\mathcal{I} = \{w \mapsto v_i w v_i^{-1} \mid 1 \leq i \leq N \text{ and } \Gamma \setminus S_i \neq \emptyset\}.$$

It is clear that \mathcal{I} generates $\text{Inn} W$. The commuting product $\prod_K \chi_{iK}$, taken over all connected components K of $\Gamma \setminus S_i$, is the inner automorphism $(w \mapsto v_i w v_i^{-1}) \in \mathcal{I}$. If, starting with \mathcal{P} , we remove one χ_{iK} for each i such that $\Gamma \setminus S_i \neq \emptyset$, then the union of the resulting set and \mathcal{I} is a generating set for $\text{Aut}^0 W$. We now do so systematically.

Definition 4.6 (\mathcal{P}^0 and $\text{Out}^0 W$). For each $1 \leq i \leq N$ such that $\Gamma \setminus S_i \neq \emptyset$, let j_i be minimal such that $v_{j_i} \in \Gamma \setminus S_i$. Define

$$\mathcal{P}^0 := \{\chi_{iK} \in \mathcal{P} \mid v_{j_i} \notin K\}.$$

Write $\text{Out}^0 W$ for the subgroup of $\text{Aut}^0 W$ generated by \mathcal{P}^0 . (In §A we write down \mathcal{P}^0 for an example.)

Remark 4.7. Observe the following properties of \mathcal{P}^0 :

1. As in Corollary 4.5, the Restricted Alphabet Rewriting Lemma may be used to show that the set $\mathcal{I} \cup \mathcal{P}^0$ is a minimal generating set for $\text{Aut}^0 W$.
2. For each $1 \leq i \leq N$, either $\Gamma \setminus S_i = \emptyset$ or the set $\mathcal{P} \setminus \mathcal{P}^0$ contains exactly one element of the form χ_{iK} .
3. If $\chi_{iK} \in \mathcal{P}^0$, then $v_1 \notin K$. If $\chi_{iK} \in \mathcal{P}^0$ and $d(v_1, v_i) \leq 1$, then $v_2 \notin K$. In general, if $\chi_{iK} \in \mathcal{P}^0$ and $d(v_j, v_i) \leq 1$ for each $1 \leq j \leq k$, then $v_{k+1} \notin K$.
4. The set \mathcal{P}^0 depends on the ordering of V defined by the indexing. For the work in this section, the ordering is unimportant.

Lemma 4.8. $\text{Out}^0 W \cap \text{Inn} W = \{id\}$.

Proof. For each $1 \leq i \leq N$, write $\mathcal{S}_i := \{\chi_{jK} \in \mathcal{P}^0 \mid v_j \in S_i\}$. Write $\mathcal{S} := \bigcap_{i=1}^N \mathcal{S}_i$. Suppose that $\phi \in \text{Inn} W \cap \text{Out}^0 W$, say $\phi(v_j) = wv_jw^{-1}$ for each $1 \leq j \leq N$. We shall use induction to show that $\phi \in \langle \mathcal{S} \rangle$.

Since $\phi \in \text{Out}^0 W$, ϕ may be written as a word Φ_0 in the alphabet $(\mathcal{P}^0)^{\pm 1}$. By Remark 4.7(3), each element of \mathcal{P}^0 acts trivially on v_1 . It follows from The Deletion Condition that w is in the centralizer of v_1 . By Lemma 2.7, $w \in W(S_1)$. By the Restricted Alphabet Rewriting Lemma, ϕ may be written as a product Φ_1 in the alphabet $\mathcal{S}_1^{\pm 1}$ (starting with Φ_0 , delete those letters not in $\mathcal{S}_1^{\pm 1}$). Now let i be an integer such that $1 \leq i < N$ and suppose that ϕ may be written as a product Φ_i in the alphabet $(\mathcal{S}_1 \cap \dots \cap \mathcal{S}_i)^{\pm 1}$. By Remark 4.7(3), each element of $\mathcal{S}_1 \cap \dots \cap \mathcal{S}_i$ acts trivially on v_{i+1} . It follows that w is in the centralizer of v_{i+1} . Hence $w \in W(S_{i+1})$. By the Restricted Alphabet Rewriting Lemma, ϕ may be written as a product Φ_{i+1} in the alphabet $(\mathcal{S}_1 \cap \dots \cap \mathcal{S}_i \cap \mathcal{S}_{i+1})^{\pm 1}$ (starting with Φ_i , delete those letters not in $\mathcal{S}_{i+1}^{\pm 1}$). By induction we have that ϕ may be written as a product Φ_N in the alphabet $\mathcal{S}^{\pm 1}$.

Now $\chi_{jK} \in \mathcal{S}$ if and only if v_j is adjacent to each vertex in Γ . But for such v_j , $\Gamma \setminus S_j = \emptyset$ and \mathcal{P} (and hence \mathcal{P}^0) contains no partial conjugations with operating letter v_j . Thus $\mathcal{S} = \emptyset$, Φ_N is the empty word and ϕ is the trivial automorphism. \square

Proof of Theorem 1.7. This follows immediately from Lemma 4.8, the fact that $\text{Inn } W$ is a normal subgroup of $\text{Aut}^0 W$ and the fact that $\mathcal{I} \sqcup \mathcal{P}^0$ generates $\text{Aut}^0 W$. \square

5 Some subgroups of $\text{Aut}^0 W$

In this section we prove Theorem 1.8, which anticipates our study of $\text{Out}^0 W$.

We first define the link points of a partial conjugation and some associated subsets of \mathcal{P} .

Definition 5.1. For a partial conjugation $\chi_{jQ} \in \mathcal{P}$ and a vertex v_i , we say that v_i is a *link point* of χ_{jQ} if $v_j \in L_i$ and $Q \cap L_i \neq \emptyset$. We write

$$\mathcal{L}_i := \{\chi_{jQ} \in \mathcal{P} \mid v_i \text{ is a link point of } \chi_{jQ}\}.$$

Example 5.2. For example, v_4 is the unique link point of $\chi_{2\{v_8, v_{15}, v_{16}\}}$ in the example examined in §A.

For an element $g \in W$, we write $\text{supp } g$ for the minimal subset of V such that g may be written as a word in the alphabet $(\text{supp } g)^{\pm 1}$ (cf. Lemma 2.3).

Proof of Theorem 1.8. Let $1 \leq i \leq N$, let Γ' be the connected component of Γ which contains v_i and let $\phi \in \langle \mathcal{L}_i \rangle$ be such that ϕ acts as the identity on $W(L_i)$. It is immediate from the definitions that $\phi(v_i) = v_i$ and $\phi(v_k) = v_k$ for each $v_k \in S_i \cup \Gamma \setminus \Gamma'$. Fix j such that $v_j \in \Gamma' \setminus S_i$. It suffices to show that $\phi(v_j) = v_j$.

Define

$$\begin{aligned} H_i^j &:= \{v_\ell \in L_i \mid \text{each path from } L_i \text{ to } v_j \text{ passes through } S_\ell\} \\ \mathcal{H}_i^j &:= \{\chi_{\ell K} \in \mathcal{L}_i \mid v_\ell \in H_i^j\}. \end{aligned}$$

Observe that if $\chi_{\ell K} \in \mathcal{H}_i^j$, then v_j and $L_i \setminus S_\ell$ are in distinct connected components of $\Gamma \setminus S_\ell$; hence $v_j \notin K$ and $\chi_{\ell K}(v_j) = v_j$. Thus $\psi(v_j) = v_j$ for each $\psi \in \langle \mathcal{H}_i^j \rangle$.

In this paragraph we prove that there exists an element $w \in \langle H_i^j \rangle$ such that $\phi(v_j) = wv_jw^{-1}$. Let $1 \leq m \leq N$ be such that $v_m \neq v_j$ and $v_m \in \text{supp } \phi(v_j)$. Since $\phi \in \langle \mathcal{L}_i \rangle$, $v_m \in L_i$. We must show that each path from L_i to v_j passes through S_m (and hence $v_m \in H_i^j$). Let $v_{j_1}, v_{j_2}, \dots, v_{j_\ell} \in V$ be the successive vertices of a path from $v_{j_1} \in L_i$ to $v_{j_\ell} = v_j$. Let $w_{j_1}, w_{j_2}, \dots, w_{j_\ell} \in$

W be minimal length words such that $\phi(v_{j_k}) = w_{j_k} v_{j_k} w_{j_k}^{-1}$. By hypothesis, ϕ acts as the identity on $W(L_i)$ and $w_{j_1} = 1$. Also by hypothesis, $v_m \in \text{supp } w_{j_\ell}$. Let k be minimal such that $v_m \notin \text{supp } w_{k-1}$ and $v_m \in \text{supp } w_k$. Since $v_{j_{k-1}}$ and v_{j_k} commute and ϕ is an automorphism, we have that $w_{j_{k-1}} v_{j_{k-1}} w_{j_{k-1}}^{-1}$ and $w_{j_k} v_{j_k} w_{j_k}^{-1}$ commute. It follows that $w_{j_k}^{-1} w_{j_{k-1}} v_{j_{k-1}} w_{j_{k-1}}^{-1} w_{j_k}$ is in the centralizer of v_{j_k} , which equals $W(S_{j_k})$ by Lemma 2.7. The following facts are consequences of the Deletion Condition:

1. $v_m \in \text{supp } w_{j_k}^{-1} w_{j_{k-1}}$;
2. if $v_m \notin \text{supp } w_{j_k}^{-1} w_{j_{k-1}} v_{j_{k-1}} w_{j_{k-1}}^{-1} w_{j_k}$, then $v_m \in S_{j_{k-1}}$;
3. if $v_m \in \text{supp } w_{j_k}^{-1} w_{j_{k-1}} v_{j_{k-1}} w_{j_{k-1}}^{-1} w_{j_k}$, then $v_m \in S_{j_k}$.

Hence $d(v_{j_{k-1}}, v_m) \leq 1$ or $d(v_{j_k}, v_m) \leq 1$ and the path $v_{j_1}, v_{j_2}, \dots, v_{j_\ell}$ passes through S_m , as required

By the paragraph above and Lemma 1.5, there exists $\psi \in \langle \mathcal{H}_i^j \rangle$ such that $\psi(v_j) = \phi(v_j)$ (note that this equality need not hold for all j). Hence $\phi(v_j) = v_j$ as required. \square

6 The group $\text{Out}^0 W$

In this section we prove Theorem 1.10 and Corollary 1.11. We first investigate the ways in which the connected components of $\Gamma \setminus S_i$ and $\Gamma \setminus S_j$ may interact.

Lemma 6.1. *Let $\chi_{iK}, \chi_{jQ} \in \mathcal{P}$. If $d(v_i, v_j) \geq 2$ and $v_j \notin K$, then $K \cap Q = \emptyset$ or $K \subset Q$.*

Proof. Assume that $d(v_i, v_j) \geq 2$, $v_j \notin K$ and $K \cap Q \neq \emptyset$. Suppose that $K \not\subset Q$. Let $v_m \in K \cap Q$, let $v_k \in K \setminus Q$ and let α be a path in K from v_m to v_k . Since $v_m \in Q$ but $v_k \notin Q$, there exists a vertex v_a on α such that $d(v_j, v_a) = 1$. Since $v_j, v_a \in \Gamma \setminus S_i$ and $d(v_j, v_a) = 1$, the vertices v_a and v_j are contained in the same connected component of $\Gamma \setminus S_i$. Hence $v_j \in K$, contradicting the hypothesis. \square

Lemma 6.2. *Let $\chi_{iK}, \chi_{jQ} \in \mathcal{P}$. If Γ is connected, then exactly one of the following thirteen cases holds:*

- (1) $d(v_i, v_j) \leq 1$;

- (2) $d(v_i, v_j) = 2, v_i \in Q, v_j \in K, K \cap Q = \emptyset;$
- (3) $d(v_i, v_j) = 2, v_i \in Q, v_j \in K, K \cap Q \neq \emptyset;$
- (4) $d(v_i, v_j) = 2, v_i \in Q, v_j \notin K, K \cap Q = \emptyset;$
- (5) $d(v_i, v_j) = 2, v_i \in Q, v_j \notin K, K \subset Q;$
- (6) $d(v_i, v_j) = 2, v_i \notin Q, v_j \in K, K \cap Q = \emptyset;$
- (7) $d(v_i, v_j) = 2, v_i \notin Q, v_j \in K, K \supset Q;$
- (8) $d(v_i, v_j) = 2, v_i \notin Q, v_j \notin K, K \cap Q = \emptyset;$
- (9) $d(v_i, v_j) = 2, v_i \notin Q, v_j \notin K, K = Q.$
- (10) $d(v_i, v_j) \geq 3, v_i \notin Q, v_j \notin K, K \cap Q = \emptyset;$
- (11) $d(v_i, v_j) \geq 3, v_i \in Q, v_j \notin K, K \subset Q;$
- (12) $d(v_i, v_j) \geq 3, v_i \notin Q, v_j \in K, K \supset Q;$
- (13) $d(v_i, v_j) \geq 3, v_i \in Q, v_j \in K, K \cup Q = \Gamma.$

The relation $\chi_{iK}\chi_{jQ} = \chi_{jQ}\chi_{iK}$ holds in cases (1), (5), (7), (8), (10), (11) and (12). The relation $\chi_{iK}\chi_{jQ} = \chi_{jQ}\chi_{iK}$ fails in cases (2), (3), (4), (6), (9) and (13).

Proof. It follows immediately from Lemma 6.1 that the cases (1)-(9) are an exhaustive list of the possibilities when $d(v_i, v_j) \leq 2$. Thus we may assume that $d(v_i, v_j) \geq 3$.

Case $v_i \notin Q, v_j \notin K$ By Lemma 6.1, either $K \cap Q = \emptyset$ or $K = Q$. Suppose that $K = Q$. Let $v_k \in K$ (hence $v_k \in Q$) be such that $d(v_i, v_k) = 2$ and let $v_{k'} \in V$ be such that $d(v_i, v_{k'}) = d(v_{k'}, v_k) = 1$. By the triangle inequality, $d(v_j, v_{k'}) \geq 2$. Since $d(v_k, v_{k'}) = 1$, v_k and $v_{k'}$ are in the same connected component of $\Gamma \setminus S_j$. Thus $v_{k'} \in Q = K$, a contradiction to the fact that $d(v_i, v_{k'}) = 1$. Hence $K \cap Q = \emptyset$.

Case $v_i \in Q$ and $v_j \notin K$ Let $v_k \in K$ be such that $d(v_i, v_k) = 2$ and let $v_{k'}$ be such that $d(v_i, v_{k'}) = d(v_{k'}, v_k) = 1$. By the triangle inequality, $d(v_j, v_{k'}) \geq 2$ and $d(v_j, v_k) \geq 1$. Since $v_j \notin K$, $d(v_j, v_k) > 1$. Since $v_i, v_{k'}, v_k \in \Gamma \setminus S_i$ and $d(v_i, v_{k'}) = d(v_{k'}, v_k) = 1$, the vertices $v_i, v_{k'}$ and

v_k are contained in the same connected component of $\Gamma \setminus S_j$. Thus $v_k \in Q$ and $K \cap Q \neq \emptyset$. By Lemma 6.1, $K \subset Q$.

Case $v_i \notin Q$ and $v_j \in K$ The proof is similar to the case $v_i \in Q$ and $v_j \notin K$ above.

Case $v_i \in Q$ and $v_j \in K$ Let v_c be a vertex in $\Gamma \setminus K$. Let α be a minimal length path from v_c to v_j . Since $v_j \in K$ and $v_c \notin K$, there exists a vertex v_a on α such that $d(v_a, v_i) \leq 1$. By the triangle inequality, $d(v_a, v_j) \geq 2$. It follows that $d(v_c, v_j) \geq 2$ also. Since $v_i, v_a, v_c \in \Gamma \setminus S_j$ and $d(v_i, v_a) \leq 1$ and the subpath of α from v_a to v_c lies in $\Gamma \setminus S_j$, the vertices v_i, v_a and v_c are contained in a single connected component of $\Gamma \setminus S_j$. Hence $v_c \in Q$ and $Q \cup K = \Gamma$.

We leave the reader to verify the statements about commuting products. \square

Remark 6.3. Assume that Γ is connected and let $\chi_{iK}, \chi_{jQ} \in \mathcal{P}^0$. It follows from the definition of \mathcal{P}^0 that $v_1 \notin K \cup Q$ and Case (13) of Lemma 6.2 is impossible. Hence if $d(v_i, v_j) \neq 2$, then the relation $\chi_{iK}\chi_{jQ} = \chi_{jQ}\chi_{iK}$ holds.

Lemma 6.4. *Let $1 \leq i < j \leq N$ be such that $d(v_i, v_j) \geq 2$ and let R be a subgraph of Γ . Then R is a connected component of $\Gamma \setminus S_i$ and $\Gamma \setminus S_j$ if and only if R is a connected component of $\Gamma \setminus (L_i \cap L_j)$ and $v_i, v_j \notin R$.*

Proof. If $d(v_i, v_j) \geq 3$, then $L_i \cap L_j = \emptyset$ and each connected component of $\Gamma \setminus (L_i \cap L_j)$ is a connected component of Γ . If R is a connected component of $\Gamma \setminus S_i$ and $\Gamma \setminus S_j$, then $v_i, v_j \notin R$ and it follows from Lemma 6.2 that R is a connected component of Γ . The result follows.

Now assume that $d(v_i, v_j) = 2$. Let Γ' denote the connected component of Γ which contains v_i and v_j . If R is not a subgraph of Γ' , then R is a connected component of $\Gamma \setminus S_i$ and $\Gamma \setminus S_j$ if and only if R is a connected component of Γ and $v_i, v_j \notin R$. The result follows. Assume that R is a connected component of $\Gamma' \setminus S_i$ and $\Gamma' \setminus S_j$. Clearly, $v_i, v_j \notin R$. Since R is a connected subgraph of $\Gamma' \setminus S_i$ and $L_i \cap L_j \subset S_i$, R is a connected subgraph of $\Gamma' \setminus (L_i \cap L_j)$. Suppose that R is not a connected component of $\Gamma' \setminus (L_i \cap L_j)$. Then there exist $v_x \in R, v_y \in \Gamma' \setminus (R \cup (L_i \cap L_j))$ such that $d(v_x, v_y) = 1$. Since $v_x \in R$ and $v_y \notin R$ and R is a connected component of $\Gamma' \setminus S_i$, $v_y \in S_i$. Similarly, $v_y \in S_j$. Thus $v_y \in S_i \cap S_j = L_i \cap L_j$ —a contradiction. Hence R is a connected component of $\Gamma' \setminus (L_i \cap L_j)$.

Now assume that $d(v_i, v_j) = 2$ and R is a connected component of $\Gamma' \setminus (L_i \cap L_j)$ and $v_i, v_j \notin R$. Since $v_i \notin R$, $S_i \cap R = \emptyset$ and R is a connected subgraph of $\Gamma' \setminus S_i$. Suppose that R is not a connected component of $\Gamma' \setminus S_i$. Then there exist $v_x \in R$, $v_y \in \Gamma' \setminus (R \cup S_i)$ such that $d(v_x, v_y) = 1$. Since $v_x \in R$ and $v_y \notin R$ and R is a connected component of $\Gamma' \setminus (L_i \cap L_j)$, $v_y \in L_i \cap L_j \subset S_i$ —a contradiction. Hence R is a connected component of $\Gamma' \setminus S_i$. Similarly, R is a connected component of $\Gamma' \setminus S_j$. \square

Recall the definition of an SIL (Definition 1.9).

Lemma 6.5. *Assume that Γ does not contain a SIL. Let $1 \leq i < j \leq N$ be such that $d(v_i, v_j) = 2$, let K_j be the connected component of $\Gamma \setminus S_i$ which contains v_j and let Q_i be the connected component of $\Gamma \setminus S_j$ which contains v_i . Then $\Gamma = K_j \cup Q_i \cup (L_i \cap L_j)$.*

Proof. Since Γ does not contain a SIL, $\Gamma \setminus (L_i \cap L_j)$ has at most two connected components. If $\Gamma \setminus (L_i \cap L_j)$ has two connected components, they are K_j and Q_i and the result is clear. Assume that $\Gamma \setminus (L_i \cap L_j)$ is connected. Let $v_x \in \Gamma \setminus (L_i \cap L_j)$ and let α be a minimal length path in $\Gamma \setminus (L_i \cap L_j)$ from v_x to v_i . If α passes through S_j , then $v_x \in K_j$. If α does not pass through S_j , then $v_x \in Q_i$. Hence the result. \square

Combined with Remark 4.1, the following lemma allows us to assume that Γ is connected when proving Theorem 1.10 and Corollary 1.11. The lemma is immediate.

Lemma 6.6. *Let Γ^+ be as in Remark 4.1. Then Γ has a SIL if and only if Γ^+ has a SIL.*

We now prove the two main results of the section.

Proof of Theorem 1.10. Assume that W is a graph product of directly-indecomposable cyclic groups. By Remark 4.1 and Lemma 6.6, we may assume without loss that Γ is connected.

Suppose Γ contains a SIL with i, j and R as in Definition 1.9. Let K denote the connected component of $\Gamma \setminus S_i$ which contains v_j and let Q denote the the connected component of $\Gamma \setminus S_j$ which contains v_i . By Lemma 6.4, $\chi_{iR}, \chi_{jR} \in \mathcal{P}$. If R does not contain the least element of $\Gamma \setminus L_i \cap L_j$, then $\chi_{iR}, \chi_{jR} \in \mathcal{P}^0$. If R does contain the least element of $\Gamma \setminus L_i \cap L_j$, then $\chi_{iK}, \chi_{jQ} \in \mathcal{P}^0$. Calculation confirms that $\chi_{iR}\chi_{jR} \neq \chi_{jR}\chi_{iR}$ and $\chi_{iK}\chi_{jQ} \neq \chi_{jQ}\chi_{iK}$. Hence $\text{Out}^0 W$ is not abelian and Property (1) implies Property (2).

Now suppose that Γ does not contain a SIL and let $\chi_{iK}, \chi_{jQ} \in \mathcal{P}^0$. By Remark 6.3, the relation $\chi_{iK}\chi_{jQ} = \chi_{jQ}\chi_{iK}$ holds whenever $d(v_i, v_j) \neq 2$. Assume that $d(v_i, v_j) = 2$. By Lemma 6.5, $\Gamma = K_j \cup Q_i \cup (L_i \cap L_j)$ for K_j and Q_i as in the statement of the Lemma. Without loss, assume that the least element of $\Gamma \setminus (L_i \cap L_j)$ is contained in K_j . By the definition of \mathcal{P}^0 , $K \neq K_j$. Thus $v_j \notin K$ and $K \subset Q_i$. If $Q = Q_i$, then $K \subset Q$ and case (5) of Lemma 6.2 holds. If $Q \neq Q_i$, then $v_i \notin Q$ and $K \cap Q = \emptyset$ and case (8) of Lemma 6.2 holds. In either case, the relation $\chi_{iK}\chi_{jQ} = \chi_{jQ}\chi_{iK}$ holds. Thus $\text{Out}^0 W$ is an abelian group and Property (2) implies Property (1). \square

Proof of Corollary 1.11. Assume that W is a graph product of primary cyclic groups. By Remark 4.1 and Lemma 6.6, we may assume without loss that Γ is connected.

Since each partial conjugation has finite order, it is clear that Property (1) implies Property (3). Suppose Γ contains a SIL with i, j and R as in Definition 1.9. Let v_r be a vertex in R . Calculation confirms that $(\chi_{iR}\chi_{jR})^n(v_r) = (v_j v_i)^n v_r (v_j v_i)^{-n}$ and $(\chi_{iR}\chi_{jR})^n(v_i) = v_i$ for each positive integer n . It follows that no power of $\chi_{iR}\chi_{jR}$ is an inner automorphism. Hence $\mathcal{O}ut W$, and $\text{Out}^0 W$, have infinite order and Property (3) implies Property (2). \square

7 The group $\text{Out}^0 W$ in the case that Γ is a tree

Theorems 1.4 and 1.7 reduce the study of $\text{Aut}^* W$ to the study of $\text{Out}^0 W$. In this section we describe the structure of $\text{Out}^0 W$ in the special case that Γ is a tree. The reader may wish to switch back and forth between this section and §A, in which we follow a specific example.

Throughout, we assume that Γ is a tree with at least three vertices. In particular, each $W(L_i)$ is a free product of cyclic groups. By reindexing, if necessary, we may further assume that indices have been assigned to elements of V so that v_1 is a leaf (that is, adjacent to exactly one vertex) and if $d(v_1, v_i) < d(v_1, v_j)$, then $i < j$.

Because Γ is a tree, each partial conjugation has a unique link point and we may define a natural partition of \mathcal{P}^0 as follows: for each $i = 1, 2, \dots, N$, define

$$\mathcal{L}_i^0 := \{\chi_{jQ} \in \mathcal{P}^0 \mid v_i \text{ is the link point of } \chi_{jQ}\}.$$

Observe the following properties:

1. $\mathcal{L}_i^0 = \mathcal{L}_i \cap \mathcal{P}^0$;
2. $\mathcal{P}^0 = \mathcal{L}_N^0 \sqcup \mathcal{L}_{N-1}^0 \sqcup \cdots \sqcup \mathcal{L}_1^0$;
3. if v_i is a leaf, then $\mathcal{L}_i^0 = \emptyset$.

Lemma 7.1. *If $\chi_{iK}, \chi_{jQ} \in \mathcal{P}^0$ are in distinct elements of the partition $\mathcal{L}_N^0 \sqcup \mathcal{L}_{N-1}^0 \sqcup \cdots \sqcup \mathcal{L}_1^0$, then χ_{iK} and χ_{jQ} commute.*

Proof. Let $\chi_{iK}, \chi_{jQ} \in \mathcal{P}^0$ be elements which do not commute. By Remark 6.3, one of cases (2), (3), (4), (6) or (9) in Lemma 6.2 must hold. We leave the reader to verify that the definition of \mathcal{P}^0 and the simple geometry of a tree imply that case (3) is impossible, and cases (2), (4), (6) and (9) may only hold if χ_{iK}, χ_{jQ} have a common link point. Thus the result. \square

Corollary 7.2. $\text{Out}^0 W = \langle \mathcal{L}_N^0 \rangle \times \langle \mathcal{L}_{N-1}^0 \rangle \times \cdots \times \langle \mathcal{L}_1^0 \rangle$.

The following proposition completes the proof of Theorem 1.12. In the statements below, we write $\mathbb{Z}_{\mathbf{m}(j)}$ for the cyclic group of order $m(j)$.

Proposition 7.3. *Suppose that Γ is a tree with at least three vertices. Let $1 \leq i \leq N$ and let $L_i = \{v_{k_1}, v_{k_2}, \dots, v_{k_M}\}$ with $k_1 < k_2 < \cdots < k_M$. If $M = 1$ (that is, v_i is a leaf) or $M > 1$ and v_{k_2} is the minimal element of $\Gamma \setminus S_{k_1}$, then*

$$\langle \mathcal{L}_i^0 \rangle \cong \text{Out}^0 W(L_i);$$

otherwise,

$$\langle \mathcal{L}_i^0 \rangle \cong \mathbb{Z}_{\mathbf{m}(k_1)} \times \text{Out}^0 W(L_i).$$

Proof. If $M = 1$, then $\mathcal{L}_i^0 = \emptyset$, $\text{Out}^0 W(L_i)$ is trivial and the result holds. So we may assume that $M > 1$. Let $\rho_i : \langle \mathcal{L}_i^0 \rangle \rightarrow \text{Aut} W(L_i)$ denote the homomorphism determined by restriction, that is, $\chi_{iK} \mapsto \chi_{iK}|_{L_i}$. Theorem 1.8 gives that ρ_i is injective. We must show that the image $\rho_i(\langle \mathcal{L}_i^0 \rangle)$ is as described in the conclusion of the Proposition.

Assume that the minimal element of $\Gamma \setminus S_{k_1}$ is v_{k_2} . Using the notation $\chi_{k_j\{k_\ell\}} := \chi_{k_j\{v_{k_\ell}\}}$, the image $\rho_i(\mathcal{L}_i^0)$ is as follows:

$$\rho_i(\mathcal{L}_i^0) = \{\chi_{k_j\{k_\ell\}} \mid 1 \leq j, \ell \leq M, j \neq \ell\} \setminus \{\chi_{k_1\{k_2\}}, \chi_{k_2\{k_1\}}, \dots, \chi_{k_M\{k_1\}}\}.$$

This is a generating set for $\text{Out}^0 W(L_i)$.

Now assume that v_{k_2} is not the minimal element of $\Gamma \setminus S_{k_1}$ (so the minimal element of $\Gamma \setminus S_{k_1}$ is not contained in L_i). The image $\rho_i(\mathcal{L}_i^0)$ is as follows:

$$\rho_i(\mathcal{L}_i^0) = \{\chi_{k_j\{k_\ell\}} \mid 1 \leq j, \ell \leq M, j \neq \ell\} \setminus \{\chi_{k_2\{k_1\}}, \dots, \chi_{k_M\{k_1\}}\}.$$

If we replace $\chi_{k_1\{k_2\}}$ by the product $\chi_{k_1\{k_2\}} \dots \chi_{k_1\{k_M\}}$, then the resulting set still generates $\langle \rho_i(\mathcal{L}_i^0) \rangle$. Observe that $\chi_{k_1\{k_2\}} \dots \chi_{k_1\{k_M\}}$ commutes with each element in the set $\rho_i(\mathcal{L}_i^0) \setminus \{\chi_{k_1\{k_2\}}\}$ and $\rho_i(\mathcal{L}_i^0) \setminus \{\chi_{k_1\{k_2\}}\}$ generates $\text{Out}^0 W(L_i)$. Thus $\rho_i(\mathcal{L}_i^0)$ generates a subgroup of $\text{Aut}^0 W(L_i)$ which is isomorphic to $\mathbb{Z}_{m(k_1)} \times \text{Out}^0 W(L_i)$. \square

Remark 7.4. Our hypotheses on the indexing of V ensure that v_1 and v_N are leaves. One may omit the corresponding terms $\text{Out}^0 W(L_1)$ and $\text{Out}^0 W(L_N)$ (and any other terms corresponding to leaves) from the statement of Theorem 1.12. However, by including these terms we ensure that the statement stays valid even if the hypotheses on the indexing is dropped.

Remark 7.5 (Presenting $\text{Aut}^* W$ in the case that Γ is a tree). Since Ab is the direct product $\prod_{k \in K} \mathbb{Z}_{m(k)}$, where K is the set

$$\{k \mid \exists i \ 1 \leq i, k \leq N, v_k \text{ is the minimal vertex in } L_i \text{ and } L_i \text{ does not contain the minimal vertex of } \Gamma \setminus S_k\},$$

one may write down a finite presentation of Ab . Since each $W(L_i)$ is a free product of cyclic groups, one may use work of Fouxé-Rabinovitch [7] (see also [20, footnote 1, p.1]) and Gilbert [8] to write down a finite presentation for $\text{Out}^0 W(L_i)$. Combining these presentations in the standard way for presenting a direct product gives a finite presentation for $\text{Out}^0 W$. Further, a finite presentation for $\text{Inn } W$ is well-known (cf. Lemma 2.5) and, because each maximal complete subgroup is a direct product of two cyclic groups, it is an easy exercise to write down a finite presentation of $\text{Aut}^1 W$. Combining the presentations of $\text{Inn } W$, $\text{Out}^0 W$ and $\text{Aut}^1 W$ in the standard way for presenting semi-direct products (including computing the image of each generator of the normal factor under conjugation by each generator of the other factor) one is then able to write down a finite presentation of $\text{Aut}^* W$ (cf. [17] [24]).

Remark 7.6. Consider the case that Γ is an arbitrary connected graph. Without loss of generality, assume that indices have been assigned to elements

of V so that if $d(v_1, v_i) < d(v_1, v_j)$, then $i < j$. Unlike the tree case, a partial conjugation may have more than one link point and the sets \mathcal{L}_i^0 do not partition \mathcal{P}^0 . However, taking inspiration from the tree case, we define a partition \mathcal{P}^0 inductively as follows: write $\mathcal{M}_1 := \mathcal{P}^0$ and for each $1 \leq i \leq N$,

$$\begin{aligned}\mathcal{L}'_i &:= \{\chi_{jQ} \in \mathcal{M}_i \mid v_i \text{ is a link point of } \chi_{jQ}\} \\ \mathcal{M}_{i+1} &:= \mathcal{M}_i \setminus \mathcal{L}'_i.\end{aligned}$$

In some cases, but not all, this partition corresponds to a semi-direct product decomposition of $\text{Out}^0 W$.

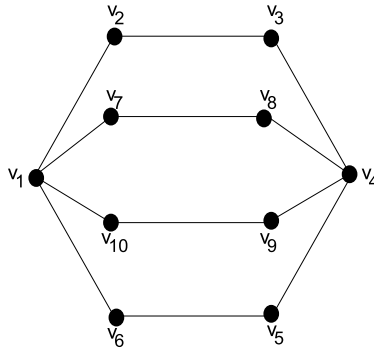
8 Applications

In this section we describe a number of applications of the results above. We begin with some applications of Corollary 1.11.

For a one ended word hyperbolic group G , $\text{Out}(G)$ is infinite if and only if G splits over a virtually cyclic subgroup with infinite center, either as an arbitrary HNN extension or as an amalgam of groups with finite center [19]. The following corollary demonstrates that such splittings are not possible in the case that W is a graph product of primary cyclic groups. The proof uses the fact that a graph product of primary cyclic groups is word hyperbolic if and only if every circuit in Γ of length four contains a chord [21] and the fact that each separating subgraph of Γ corresponds to a splitting of W as a free product with amalgamation (with the separating subgraph generating the amalgamated subgroup).

Corollary 8.1. *If W is a graph product of primary cyclic groups and W is a one ended word hyperbolic group, then $\text{Out } W$ is finite.*

Proof. Let W be a graph product of primary cyclic groups which is one ended and word hyperbolic. Suppose that Γ contains a SIL. By Theorem 1.11, there exist i, j, R as in Definition 1.9. If $L_i \cap L_j$ is a complete graph, then $W(L_i \cap L_j)$ is finite and it follows from the Ends Theorem of Hopf and Stallings (see, for example, [2, Theorem I.8.32]) that W has infinitely many ends—a contradiction to the hypothesis. Thus $L_i \cap L_j$ is not a complete subgraph and there exist non-adjacent vertices $v_x, v_y \in L_i \cap L_j$. Then $v_i v_x v_j v_y$ is a non-chordal square and W is not word hyperbolic—again, a contradiction to the hypothesis. \square

Figure 2: The graph Γ for Remark 8.2.

Remark 8.2. In [23], the authors construct one-ended hyperbolic groups with finite outer automorphism group and a non-trivial JSJ decomposition in the sense of Bowditch (that is, the group has a non-trivial graph of groups decomposition with two-ended edge groups and vertex groups which are either two-ended, maximal “hanging fuchsian”, or non-elementary quasiconvex subgroups not of the previous two types—for more details, see [1]). Such groups necessarily have only the trivial JSJ decomposition in the sense of Sela since the outer automorphism groups are finite. Using Corollary 1.11, one may construct examples of right-angled Coxeter groups with similar properties to the groups described in [23]. In particular, if W is a right-angled Coxeter group and the following properties hold:

1. $\Gamma \setminus \Delta$ is connected for each complete subgraph Δ ;
2. every circuit in Γ of length four contains a chord;
3. $\Gamma \setminus \Lambda$ is disconnected for some subgraph Λ which generates a virtually abelian group;
4. Γ has no SIL;

then W is a one-ended hyperbolic group with a non-trivial JSJ decomposition in the sense of Bowditch and $\mathcal{O}ut W$ is finite. For example, the graph Γ of Figure 2 has the desired properties (with $\Lambda = \{v_1, v_4\}$).

If W is a graph product of primary cyclic groups, then there exists a geometric action of W on a CAT(0) space [21]. We say that W has *isolated flats* if there exists a geometric action of W on a CAT(0) space with isolated

flats (see [14]). To prove the lemma below we shall need only the following property of such groups, which follows from the results in [14]:

- (*) if W has isolated flats and $S_1, S_2 \subseteq W$ are subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and $S_1 \cap S_2 \neq \{id\}$, then $\langle S_1, S_2 \rangle$ is virtually-abelian.

Lemma 8.3. *If W is a graph product of primary cyclic groups and W is one ended with isolated flats and $1 \leq i < j \leq N$ are such that $d(v_i, v_j) = 2$, then $W(L_i \cap L_j)$ is virtually abelian.*

Proof. Let W , i and j be as in the hypothesis of the lemma. Suppose that $W(L_i \cap L_j)$ is not virtually abelian. Graph products of primary cyclic groups are subgroups of Coxeter groups [15, Corollary 5.11] and hence linear and satisfy the Tits Alternative. Further, since W acts geometrically on a CAT(0) space, each virtually solvable subgroup is virtually abelian [2, p. 249]. It follows that there exist elements $a, b \in W(L_i \cap L_j)$ such that $\langle a, b \rangle$ is a free group of rank two. The subgroups $S_1 = \langle v_i v_j, a \rangle$ and $S_2 = \langle v_i v_j, b \rangle$ witness that W does not have property (*), since $\langle S_1 \cup S_2 \rangle$ contains the subgroup $\langle a, b \rangle$. \square

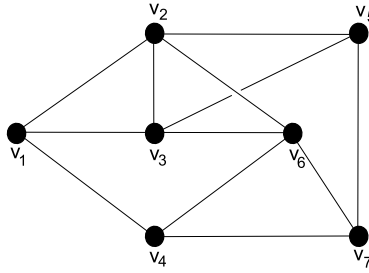
Corollary 8.4. *If W is a graph product of primary cyclic groups and W is one ended with isolated flats and $\mathcal{O}ut W$ is infinite, then W splits as a free product with amalgamation $W = A *_C B$ where*

1. A and B are special subgroups; and
2. C is an infinite virtually abelian special subgroup.

Proof of Corollary 8.4. Let W and $\mathcal{O}ut W$ be as in the hypothesis of the corollary. By Theorem 1.11, there exist $1 \leq i < j \leq N$ such that $d(v_i, v_j) = 2$ and $L_i \cap L_j$ separates Γ . By Lemma 8.3, $W(L_i \cap L_j)$ is virtually abelian. Since W is one-ended, $W(L_i \cap L_j)$ is not finite. The result follows, with $C = L_i \cap L_j$. \square

We say that W has *property (NLC)* if for every CAT(0) space X on which W acts geometrically, the visual boundary ∂X (see [2, p. 264]) is not locally connected.

Corollary 8.5. *If W is a right-angled Coxeter group and $\mathcal{O}ut W$ is infinite, then W has property (NLC).*

Figure 3: The graph Γ for Remark 8.6.

Proof. Assume that $\text{Out } W$ is infinite. By Theorem 1.11, there exist i, j, R as in Definition 1.9. It follows that W is not finite or two-ended. If W has infinitely-many ends, then W has property (NLC). Assume that W is one ended. Since $(L_i \cap L_j, L_i \cap L_j, \{v_i, v_j\})$ is a ‘virtual factor separator’ [22, Definition 3.1] and $L_i \cap L_j$ is not a ‘suspended separator’ [22, Definition 3.1], we may apply [22, Theorem 3.2(2)] to conclude that W has property (NLC). \square

Remark 8.6. We now demonstrate that the converse to Corollary 8.5 does not hold. Let W be the right-angled Coxeter group corresponding to the graph Γ in Figure 3. Observe that Γ does not contain a SIL, but $(\{v_2, v_3, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_6\})$ is a virtual factor separator and $\{v_2, v_3, v_4\}$ is not a suspended separator. Thus $\text{Out } W$ is finite, by Theorem 1.11, and W has property (NLC), by [22, Theorem 3.2(2)].

The following application offers a glimpse of some geometry of $\text{Aut } W$.

Corollary 8.7. *Let W be a graph product of primary cyclic groups. Then $\text{Aut } W$ is word hyperbolic if and only if the following conditions are satisfied:*

1. Γ has no SIL;
2. every circuit in Γ of length four contains a chord.

Proof. Assume that Γ has an SIL. Let i, j, R be as in Definition 1.9 and let $\iota_{v_i v_j}$ denote the inner automorphism $w \mapsto v_i v_j w v_j^{-1} v_i^{-1}$. Since $d(v_i, v_j) \geq 2$, $\iota_{v_i v_j}$ has infinite order. The product $\chi_{jR} \chi_{iR}$ also has infinite order and $\langle \iota_{v_i v_j}, (\chi_{jR} \chi_{iR}) \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Thus $\text{Aut } W$ is not hyperbolic.

Assume that Γ has no SIL. By Corollary 1.11, $\text{Out } W$ is finite and $\text{Inn } W$ is a finite-index subgroup of $\text{Aut } W$. But $\text{Inn } W$ is also a finite-index subgroup of W (see §2). Thus $\text{Aut } W$ and W are commensurable, and hence

quasi-isometric (see [2, Example I.8.8.20(1)]). The result follows immediately from the characterization of word hyperbolic graph products of primary cyclic groups described above and the fact that word-hyperbolicity is a quasi-isometry invariant [2, Theorem III.H.1.9]. \square

The authors of [10] give sufficient conditions (distinct from those below) for $\text{Aut } W$ to split as $\text{Inn } W \rtimes \text{Out } W$ in the case that W is a right-angled Coxeter group. They then describe an application of their results to group extensions. We now follow an analogous program for graph products of primary cyclic groups. For each $1 \leq i \leq N$, we write $\Delta_i := \{v_j \in V \mid S_i = S_j\}$ (note that Δ_i is a complete subgraph for each $1 \leq i \leq N$).

Lemma 8.8. *If W is a graph product of directly-indecomposable cyclic groups and $\phi(W(\Delta_i)) = W(\Delta_i)$ for each $1 \leq i \leq N$ and $\phi \in \text{Aut}^1 W$, then the splittings of Theorems 1.4 and 1.7 are compatible; that is, one may write*

$$\text{Aut}^* W = \text{Inn } W \rtimes (\text{Out}^0 W \rtimes \text{Aut}^1 W) \cong \text{Inn } W \rtimes \text{Out}^* W.$$

Proof. Let $\phi \in \text{Aut}^1 W$ and $\theta \in \text{Out}^0 W$. If θ is not the identity, then the product $\phi\theta$ is not an inner automorphism since it acts non-trivially on the set of conjugacy classes of cyclically reduced involutions in W . If θ is the identity but ϕ is not, then the product $\phi\theta = \phi$ is not an inner automorphism by Lemma 4.8. Thus, to show that $\text{Inn } W \cap \text{Out}^* W = \{id\}$ and hence the result, it suffices to show that $\text{Out}^* W = \text{Out}^0 W \cdot \text{Aut}^1 W$.

Let $\phi \in \text{Aut}^1 W$ and $\chi_{iK} \in \mathcal{P}^0$. It suffices to show that $\phi^{-1}\chi_{iK}\phi \in \langle \mathcal{P}^0 \rangle$. Let $1 \leq j \leq N$. If $v_j \notin K$, then $\Delta_j \cap K = \emptyset$ and the support of $\phi(v_j)$ is disjoint from K . Hence

$$\phi^{-1}\chi_{iK}\phi(v_j) = \phi^{-1}(\chi_{iK}(\phi(v_j))) = \phi^{-1}\phi(v_j) = v_j.$$

If $v_j \in K$, then $\Delta_j \subseteq K$ and the support of $\phi(v_j)$ is contained in K . Hence

$$\phi^{-1}\chi_{iK}\phi(v_j) = \phi^{-1}(\chi_{iK}(\phi(v_j))) = \phi^{-1}(v_i\phi(v_j)v_i^{-1}) = \phi^{-1}(v_i)v_j(\phi^{-1}(v_i))^{-1}.$$

Thus we have

$$\phi^{-1}\chi_{iK}\phi(v_j) = \begin{cases} v_j & \text{if } v_j \notin K, \\ \phi^{-1}(v_i)v_j(\phi^{-1}(v_i))^{-1} & \text{if } v_j \in K. \end{cases}$$

By hypothesis, $\phi^{-1}(v_i) \in W(\Delta_i)$. For each $v_\ell \in \Delta_i$, the least element of $\Gamma \setminus S_i$ is also the least element of $\Gamma \setminus S_\ell$ and, since $\chi_{iK} \in \mathcal{P}^0$, we have $\chi_{\ell K} \in \mathcal{P}^0$. Thus $\phi^{-1}\chi_{iK}\phi$ may be written as a product of elements in $(\mathcal{P}^0)^{\pm 1}$. \square

Recall that the center of W is the special subgroup generated by those vertices adjacent to every other vertex. Recall also that Lemma 1.2 gives some sufficient conditions for the equality $\text{Aut}^* W = \text{Aut} W$.

Corollary 8.9. *If W is a graph product of directly-indecomposable cyclic groups and the following conditions are satisfied:*

1. W has trivial center;
2. $\text{Aut}^* W = \text{Aut} W$;
3. $\phi(W(\Delta_i)) = W(\Delta_i)$ for each $1 \leq i \leq N$ and each $\phi \in \text{Aut}^1 W$;

then all extensions

$$1 \rightarrow W \rightarrow E \rightarrow G \rightarrow 1$$

are trivial (that is, split extensions).

Proof. Conditions (2) and (3) and Lemma 8.8 give that $\text{Aut} W = \text{Inn} W \rtimes \text{Out} W$. Thus each homomorphism $\psi : G \rightarrow \text{Out} W$ lifts to a homomorphism $\hat{\psi} : G \rightarrow \text{Aut} W$ and hence determines a semidirect product $W \rtimes_{\hat{\psi}} G$. Condition (1) of the hypothesis ensures that there is exactly one extension of G by W (up to equivalence) corresponding to any homomorphism $\psi : G \rightarrow \text{Out} W$ [3, Corollary IV.6.8 p.104]. \square

In the case that W is a right-angled Coxeter group, we may express the hypotheses of Corollary 8.9 entirely in terms of the labeled-graph Γ .

Corollary 8.10. *If W is a right-angled Coxeter group and the following conditions are satisfied:*

1. $\Gamma \setminus S_i \neq \emptyset$ for each $1 \leq i \leq N$;
2. $f(\Delta_i) = \Delta_i$ for each labeled-graph isomorphism $f : (\Gamma, \mathbf{m}) \rightarrow (\Gamma, \mathbf{m})$ and each $1 \leq i \leq N$;
3. $S_i \subseteq S_j$ if and only if $S_i = S_j$ for each $1 \leq i, j \leq N$;

then all extensions

$$1 \rightarrow W \rightarrow E \rightarrow G \rightarrow 1$$

are trivial (that is, split extensions).

Proof. Condition (1) implies that W has trivial center. By Lemma 1.2(1), $\text{Aut}^* W = \text{Aut} W$. Since W is a right-angled Coxeter group, the group $\text{Aut}^1 W$ is generated by those automorphisms induced by symmetries of Γ and by automorphisms of the form

$$v_i \mapsto v_i v_j, \quad v_k \mapsto v_k \text{ for each } k \neq i,$$

for some $1 \leq i, j \leq N$ for which $i \neq j$ and $S_i \subseteq S_j$. It follows that Conditions (2) and (3) imply that $\phi(W(\Delta_i)) = W(\Delta_i)$ for each $1 \leq i \leq N$ and each $\phi \in \text{Aut}^1 W$. \square

Remark 8.11. We conjecture that, for an arbitrary graph product of directly-indecomposable cyclic groups W , the analogue of Corollary 8.10 is true provided we replace Condition (3) by the following:

(3') if $S_i \subseteq S_j$ and either $\mathbf{m}(j)$ divides $\mathbf{m}(i)$ or $\mathbf{m}(i) = \infty$, then $S_i = S_j$.

We now consider some applications of Theorem 1.12. Recall that we write V (resp. E) for the set of vertices (resp. edges) of Γ and $N = |V|$. Let $V_1 \subset V$ denote the set of vertices which have valence one (the ‘leaves’ of Γ).

Corollary 8.12. *If W is a graph product of primary cyclic groups and Γ is a tree, then $\text{Out} W$ is virtually torsion-free and*

$$\text{vcd}(\text{Out} W) = |V_1| - 2.$$

Proof. It follows from Lemma 1.2(1) and Theorem 1.12 that the product $\prod_{i=1}^N \text{Out}^0 W(L_i)$ is isomorphic to a subgroup of finite index in $\text{Out} W$. Thus it suffices to calculate the virtual cohomological dimension of this product.

For each i , $W(L_i)$ is a free product of finite groups and so $\text{Out}^0 W(L_i)$ is virtually torsion-free [6] and $\text{vcd}(\text{Out}^0 W(L_i)) = \max\{0, |L_i| - 2\}$ [16] [20, p. 67]. The direct product of virtually torsion-free groups is virtually torsion-free, so $\prod_{i=1}^N \text{Out}^0 W(L_i)$ is virtually torsion-free.

It follows from [3, Proposition VII.2.4(b) p.187] (see also [3, VII.11 exercise 2 p.229]) that the virtual cohomological dimension of a direct product is at most the sum of the virtual cohomological dimensions of the factors. Thus we have

$$\text{vcd}\left(\prod_{i=1}^N \text{Out}^0 W(L_i)\right) \leq \sum_{i=1}^N \max\{0, |L_i| - 2\}.$$

For each $1 \leq i \leq N$, $\text{Out}^0 W(L_i)$ contains a free abelian subgroup of rank $\max\{0, |L_i| - 2\}$ (if $L_i = \{v_{j_1}, \dots, v_{j_M}\}$, then $\{(\chi_{j_2\{j_3\}}\chi_{j_1\{j_3\}}), \dots, (\chi_{j_2\{j_M\}}\chi_{j_1\{j_M\}})\}$ generates a free abelian subgroup). It follows that the product $\prod_{i=1}^N \text{Out}^0 W(L_i)$ contains a free abelian subgroup of rank

$$\sum_{i=1}^N \max\{0, |L_i| - 2\}$$

and

$$\text{vcd}\left(\prod_{i=1}^N \text{Out}^0 W(L_i)\right) = \sum_{i=1}^N \max\{0, |L_i| - 2\}.$$

Finally,

$$\begin{aligned} \sum_{i=1}^N \max\{0, |L_i| - 2\} &= \left(\sum_{i=1}^N (|L_i| - 2)\right) + |V_1| \\ &= \left(\sum_{i=1}^N |L_i|\right) - 2N + |V_1| \\ &= 2|E| - 2|V| + |V_1| \\ &= 2|E| - 2(|E| + 1) + |V_1| \\ &= |V_1| - 2. \end{aligned}$$

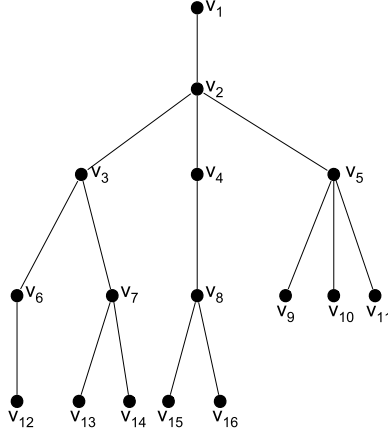
(the first equality holds because $|L_i| - 2 < 0$ if and only if $v_i \in V_1$ and $|L_i| - 2 = -1$, the third equality holds because each edge in Γ contributes to $|L_i|$ for two values of i and the fourth equality holds because $|V| = |E| + 1$). \square

The following corollary extends the main results from [11].

Corollary 8.13. *If W is a right-angled Artin group and Γ is a tree, then there exist regular languages of normal forms for $\text{Out}^0 W$ and $\text{Aut}^0 W$.*

Proof. Consider the structure of $\text{Out}^0 W$ as described in Theorem 1.12. For each $1 \leq i \leq N$, $W(L_i)$ is a free group and there exists a regular language of normal forms \mathcal{N}_i for $\text{Out}^0 W(L_i)$ [11]. Since Ab is a finitely-generated free abelian group, there is a regular language of normal forms \mathcal{N}_{Ab} for Ab . The language $\mathcal{N}_{\text{Ab}}\mathcal{N}_1\mathcal{N}_2 \dots \mathcal{N}_N$ is a regular language of normal forms for $\text{Out}^0 W$.

Further, $\text{Inn} W$ is a right-angled Artin group and hence is automatic [13, Theorem B]. It follows that there is a regular language of normal forms \mathcal{N}_I for $\text{Inn} W$. By Theorem 1.7, the language $\mathcal{N}_I\mathcal{N}_{\text{Ab}}\mathcal{N}_1\mathcal{N}_2 \dots \mathcal{N}_N$ is a regular language of normal forms for $\text{Aut}^0 W$. \square

Figure 4: The graph Γ for §A.

A An example

Let Γ be the tree depicted in Figure 4 and \mathbf{m} an order map on Γ . By Laurence's result [17, Theorem 4.1], $\text{Aut}^0 W$ is generated by the set \mathcal{P} :

$$\begin{aligned}
 \mathcal{P} = & \{ \chi_1\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_1\{v_4, v_8, v_{15}, v_{16}\}, \chi_1\{v_5, v_9, v_{10}, v_{11}\}, \\
 & \chi_2\{v_6, v_{12}\}, \chi_2\{v_7, v_{13}, v_{14}\}, \chi_2\{v_8, v_{15}, v_{16}\}, \chi_2\{v_9\}, \chi_2\{v_{10}\}, \chi_2\{v_{11}\}, \\
 & \chi_3\{v_1\}, \chi_3\{v_{12}\}, \chi_3\{v_{13}\}, \chi_3\{v_{14}\}, \chi_3\{v_4, v_8, v_{15}, v_{16}\}, \chi_3\{v_5, v_9, v_{10}, v_{11}\}, \\
 & \chi_4\{v_1\}, \chi_4\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_4\{v_{15}\}, \chi_4\{v_{16}\}, \chi_4\{v_5, v_9, v_{10}, v_{11}\}, \\
 & \chi_5\{v_1\}, \chi_5\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_5\{v_4, v_8, v_{15}, v_{16}\}, \\
 & \chi_6\{v_1, v_2, v_4, v_5, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}\}, \chi_6\{v_7, v_{13}, v_{14}\}, \\
 & \chi_7\{v_1, v_2, v_4, v_5, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}\}, \chi_7\{v_6, v_{12}\}, \\
 & \chi_8\{v_1, v_2, v_3, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}, \\
 & \chi_9\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_9\{v_{10}\}, \chi_9\{v_{11}\}, \\
 & \chi_{10}\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_{10}\{v_9\}, \chi_{10}\{v_{11}\}, \\
 & \chi_{11}\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_{11}\{v_9\}, \chi_{11}\{v_{10}\}, \\
 & \chi_{12}\{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}, v_{16}\}, \\
 & \chi_{13}\{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}\}, \chi_{13}\{v_{14}\}, \\
 & \chi_{14}\{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}\}, \chi_{14}\{v_{13}\}, \\
 & \chi_{15}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}, \chi_{15}\{v_{16}\}, \\
 & \chi_{16}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}, \chi_{16}\{v_{15}\} \}.
 \end{aligned}$$

The sets $\mathcal{L}_1, \dots, \mathcal{L}_{16}$ are as follows:

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{L}_9 = \mathcal{L}_{10} = \mathcal{L}_{11} = \mathcal{L}_{12} = \mathcal{L}_{13} = \mathcal{L}_{14} = \mathcal{L}_{15} = \mathcal{L}_{16} = \emptyset \\
\mathcal{L}_2 &= \{ \chi_1\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_1\{v_4, v_8, v_{15}, v_{16}\}, \chi_1\{v_5, v_9, v_{10}, v_{11}\}, \chi_3\{v_1\}, \chi_3\{v_4, v_8, v_{15}, v_{16}\}, \\
&\quad \chi_3\{v_5, v_9, v_{10}, v_{11}\}, \chi_4\{v_1\}, \chi_4\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_4\{v_5, v_9, v_{10}, v_{11}\}, \chi_5\{v_1\}, \\
&\quad \chi_5\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_5\{v_4, v_8, v_{15}, v_{16}\} \} \\
\mathcal{L}_3 &= \{ \chi_2\{v_6, v_{12}\}, \chi_2\{v_7, v_{13}, v_{14}\}, \chi_6\{v_1, v_2, v_4, v_5, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}\}, \chi_6\{v_7, v_{13}, v_{14}\}, \\
&\quad \chi_7\{v_1, v_2, v_4, v_5, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}\}, \chi_7\{v_6, v_{12}\} \} \\
\mathcal{L}_4 &= \{ \chi_2\{v_8, v_{15}, v_{16}\}, \chi_8\{v_1, v_2, v_3, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\} \} \\
\mathcal{L}_5 &= \{ v_2\{v_9\}, v_2\{v_{10}\}, v_2\{v_{11}\}, \chi_9\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_9\{v_{10}\}, \chi_9\{v_{11}\}, \\
&\quad \chi_{10}\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_{10}\{v_9\}, \chi_{10}\{v_{11}\}, \\
&\quad \chi_{11}\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}, \chi_{11}\{v_9\}, \chi_{11}\{v_{10}\} \} \\
\mathcal{L}_6 &= \{ \chi_3\{v_{12}\}, \chi_{12}\{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}, v_{16}\} \} \\
\mathcal{L}_7 &= \{ \chi_3\{v_{13}\}, \chi_3\{v_{14}\}, \chi_{13}\{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}\}, \chi_{13}\{v_{14}\}, \\
&\quad \chi_{14}\{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}\}, \chi_{14}\{v_{13}\} \} \\
\mathcal{L}_8 &= \{ \chi_4\{v_{15}\}, \chi_4\{v_{16}\}, \chi_{15}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}, \chi_{15}\{v_{16}\}, \\
&\quad \chi_{16}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}, \chi_{16}\{v_{15}\} \}.
\end{aligned}$$

To construct \mathcal{P}^0 from \mathcal{P} , we remove the first partial conjugation from each line in the description of \mathcal{P} above to get:

$$\begin{aligned}
\mathcal{P}^0 &= \{ \chi_1\{v_4, v_8, v_{15}, v_{16}\}, \chi_1\{v_5, v_9, v_{10}, v_{11}\}, \chi_2\{v_7, v_{13}, v_{14}\}, \chi_2\{v_8, v_{15}, v_{16}\}, v_2\{v_9\}, v_2\{v_{10}\}, \\
&\quad v_2\{v_{11}\}, \chi_3\{v_{12}\}, \chi_3\{v_{13}\}, \chi_3\{v_{14}\}, \chi_3\{v_4, v_8, v_{15}, v_{16}\}, \chi_3\{v_5, v_9, v_{10}, v_{11}\}, \\
&\quad \chi_4\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_4\{v_{15}\}, \chi_4\{v_{16}\}, \chi_4\{v_5, v_9, v_{10}, v_{11}\}, \chi_5\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \\
&\quad \chi_5\{v_4, v_8, v_{15}, v_{16}\}, \chi_6\{v_7, v_{13}, v_{14}\}, \chi_7\{v_6, v_{12}\}, \chi_9\{v_{10}\}, \chi_9\{v_{11}\}, \chi_{10}\{v_9\}, \chi_{10}\{v_{11}\}, \\
&\quad \chi_{11}\{v_9\}, \chi_{11}\{v_{10}\}, \chi_{13}\{v_{14}\}, \chi_{14}\{v_{13}\}, \chi_{15}\{v_{16}\}, \chi_{16}\{v_{15}\} \}.
\end{aligned}$$

The groups $\langle \mathcal{L}_1^0 \rangle, \dots, \langle \mathcal{L}_{16}^0 \rangle$ are as follows:

$$\langle \mathcal{L}_i^0 \rangle \cong \langle \emptyset \rangle \cong \{id\} \cong \text{Out } W(L_i) \text{ for } i = 1, 9, 10, 11, 12, 13, 14, 15, 16;$$

$$\begin{aligned}
\langle \mathcal{L}_2^0 \rangle &\cong \langle \{ \chi_1\{v_4, v_8, v_{15}, v_{16}\}, \chi_1\{v_5, v_9, v_{10}, v_{11}\}, \chi_3\{v_4, v_8, v_{15}, v_{16}\}, \chi_3\{v_5, v_9, v_{10}, v_{11}\}, \\
&\quad \chi_4\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_4\{v_5, v_9, v_{10}, v_{11}\}, \chi_5\{v_3, v_6, v_7, v_{12}, v_{13}, v_{14}\}, \chi_5\{v_4, v_8, v_{15}, v_{16}\} \} \rangle \\
&\cong \text{Out}^0 W(L_2);
\end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_3^0 \rangle &\cong \langle \{\chi_{2\{v_7, v_{13}, v_{14}\}}, \chi_{6\{v_7, v_{13}, v_{14}\}}, \chi_{7\{v_6, v_{12}\}}\} \rangle \\ &\cong \text{Out}^0 W(L_3); \end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_4^0 \rangle &\cong \langle \{\chi_{2\{v_8, v_{15}, v_{16}\}}\} \rangle \\ &\cong \mathbb{Z}_{\mathbf{m}(2)} \\ &\cong \mathbb{Z}_{\mathbf{m}(2)} \times \text{Out}^0 W(L_4); \end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_5^0 \rangle &\cong \langle \{v_{2\{v_9\}}, v_{2\{v_{10}\}}, v_{2\{v_{11}\}}, \chi_{9\{v_{10}\}}, \chi_{9\{v_{11}\}}, \chi_{10\{v_9\}}, \chi_{10\{v_{11}\}}, \chi_{11\{v_9\}}, \chi_{11\{v_{10}\}}\} \rangle \\ &\cong \langle \{v_{2\{v_9\}}\} \cup \{v_{2\{v_{10}\}}, v_{2\{v_{11}\}}, \chi_{9\{v_{10}\}}, \chi_{9\{v_{11}\}}, \chi_{10\{v_9\}}, \chi_{10\{v_{11}\}}, \chi_{11\{v_9\}}, \chi_{11\{v_{10}\}}\} \rangle \\ &\cong \langle \{v_{2\{v_9, v_{10}, v_{11}\}}\} \cup \{v_{2\{v_{10}\}}, v_{2\{v_{11}\}}, \chi_{9\{v_{10}\}}, \chi_{9\{v_{11}\}}, \chi_{10\{v_9\}}, \chi_{10\{v_{11}\}}, \chi_{11\{v_9\}}, \chi_{11\{v_{10}\}}\} \rangle \\ &\cong \mathbb{Z}_{\mathbf{m}(2)} \times \text{Out}^0 W(L_5); \end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_6^0 \rangle &\cong \langle \{\chi_{3\{v_{12}\}}\} \rangle \\ &\cong \mathbb{Z}_{\mathbf{m}(3)} \\ &\cong \mathbb{Z}_{\mathbf{m}(3)} \times \text{Out}^0 W(L_6); \end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_7^0 \rangle &\cong \langle \{\chi_{3\{v_{13}\}}, \chi_{3\{v_{14}\}}, \chi_{13\{v_{14}\}}, \chi_{14\{v_{13}\}}\} \rangle \\ &\cong \langle \{\chi_{3\{v_{13}\}}\} \cup \{\chi_{3\{v_{14}\}}, \chi_{13\{v_{14}\}}, \chi_{14\{v_{13}\}}\} \rangle \\ &\cong \langle \{\chi_{3\{v_{13}, v_{14}\}}\} \cup \{\chi_{3\{v_{14}\}}, \chi_{13\{v_{14}\}}, \chi_{14\{v_{13}\}}\} \rangle \\ &\cong \mathbb{Z}_{\mathbf{m}(3)} \times \text{Out}^0 W(L_7); \end{aligned}$$

$$\begin{aligned}\langle \mathcal{L}_8^0 \rangle &\cong \langle \{\chi_{4\{v_{15}\}}, \chi_{4\{v_{16}\}}, \chi_{15\{v_{16}\}}, \chi_{16\{v_{15}\}}\} \rangle \\ &\cong \langle \{\chi_{4\{v_{15}\}}\} \cup \{\chi_{4\{v_{16}\}}, \chi_{15\{v_{16}\}}, \chi_{16\{v_{15}\}}\} \rangle \\ &\cong \langle \{\chi_{4\{v_{15}, v_{16}\}}\} \cup \{\chi_{4\{v_{16}\}}, \chi_{15\{v_{16}\}}, \chi_{16\{v_{15}\}}\} \rangle \\ &\cong \mathbb{Z}_{\mathbf{m}(4)} \times \text{Out}^0 W(L_8). \end{aligned}$$

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