

BOUNDARIES OF $CAT(0)$ GROUPS OF THE FORM $\Gamma = G \times H$

KIM E. RUANE

ABSTRACT. Boundaries of groups which admit geometric actions on $CAT(0)$ spaces are not yet well-defined because the homeomorphism question of Gromov's has not been answered. This question, as well as the Tits metric question is answered affirmatively for $CAT(0)$ groups of the form $G \times H$ where both factors are non-elementary word hyperbolic groups.

0. INTRODUCTION

Just as the word hyperbolic groups provide a generalization of the classical hyperbolic groups, finitely generated free groups, and certain small cancellation groups, there should be a general class of nonpositively curved groups that includes finitely generated free abelian groups, fundamental groups of compact nonpositively curved Riemannian manifolds, and more general small cancellation groups. In [Gr], Gromov proposes such a class by defining a group to be nonpositively curved if it acts geometrically on a nonpositively curved geometry. An isometric action of a discrete group G on a metric space is *geometric* if it is properly discontinuous and cocompact. One problem in extending this theory is to find the correct notion of nonpositive curvature. One such notion is provided by the work of Alexandrov and Toponogov [A], wherein one compares triangles in a given geometry X with triangles of the same size in \mathbb{E}^2 and asks that the triangles in X be at least as thin as those the comparison triangles in \mathbb{E}^2 (rigorous definitions are provided in section 1). These " $CAT(0)$ " spaces have many of the same geometric properties as universal covers of compact Riemannian manifolds (in fact, all such universal covers are $CAT(0)$ spaces) including convexity of the distance function, unique geodesic segments, and contractibility. These spaces come equipped with a natural compactification by adding on the geodesic rays emanating from a point to obtain a compact metrizable space just as in the δ -negatively curved setting. This boundary is our main object of study and definitions and basic properties of this boundary are given in section 1.

For δ -negatively curved spaces, this notion of boundary is well-defined in the sense that if G acts geometrically on two geometries, their boundaries are homeomorphic via a G -equivariant homeomorphism coming from the orbit maps, thus the notation ∂G is unambiguous. The guiding question for this paper was provided by Gromov, [Gr] and asks whether this same well-definedness holds in the nonpositively curved setting.

1991 *Mathematics Subject Classification*. Primary: 20F32; Secondary: 20H15.

Key words and phrases. word hyperbolic groups, $CAT(0)$ spaces, boundary of groups, Tits metric.

Question. *When a group Γ acts geometrically on two $CAT(0)$ spaces, are their visual boundaries Γ -equivariantly homeomorphic? Are their boundaries isometric in the Tits metric?*

This paper is organized into five sections. The first contains some basic facts about $CAT(0)$ spaces and their boundaries. We do not provide proofs of these facts, but references are given. The second section provides a brief review of the Tits metric as we use this in the proof of the Main Theorem stated below. The proof will be the contents of section three. In the remaining two sections we provide a proof of the Tits metric question which is also proven in [BH], a proof that the periodic one flats and two flats are dense among all flats in this case, and finally some open questions.

Main Theorem. *Suppose $\Gamma = G \times H$ acts geometrically on a $CAT(0)$ space X with G, H non-elementary word hyperbolic. Then the following are true:*

- (1) *There exist closed convex subsets X_1, X_2 of X such that G and H admit geometric actions on X_1 and X_2 respectively. Furthermore, $L(X_1) \cap L(X_2) = \emptyset$ in ∂X , where $L(X_i)$ denotes the limit set of X_i , giving $\partial X \cong \partial G \star \partial H$.*
- (2) *Both G and H determine quasi-convex subsets of X - i.e. for a basepoint $x_0 \in X$, the orbits $(G \times \{1_H\}) \cdot x_0$ and $(\{1_G\} \times H) \cdot x_0$ are quasi-convex subsets of X .*

In [B], the following related result is shown to be true where the hypotheses are as above.

Theorem. *If Γ has trivial center, acts freely on X , and if X satisfies the local extension property for geodesics, then X splits as a product and the action **MUST** be the product action.*

There are theorems in the literature which are similar to the theorem stated above. The result for Riemannian manifolds is the classical splitting theorem of [GW] which says if the fundamental group of the manifold splits as a product, then so does the manifold, it's universal cover, and the action. The theorem stated above from [B] provides a generalization of this to general $CAT(0)$ spaces. Both of these theorems have a group acting as a group of covering transformations on a $CAT(0)$ space and both need some added conditions to guarantee the full strength that the action splits as the product action. Our theorem is not asking for this much structure, only an "asymptotic" splitting - i.e. we want the boundary of the $CAT(0)$ space to split as the join - we don't need the action to be free nor do we need local extendability of geodesics, nor do we need a trivial center to get our result.

It is also worth mentioning the following theorem for comparison:

Theorem - [BR11]. *Whenever $\Gamma = G \times \mathbb{Z}^n$ where G is word hyperbolic acts geometrically on the $CAT(0)$ space X , there is an embedding $\partial G \rightarrow \partial X$ that extends to a homeomorphism of $\partial G \star S^{n-1}$ onto ∂X . Moreover, if Γ also acts geometrically on the $CAT(0)$ space X' , then there is a Γ -equivariant homeomorphism $\partial X \rightarrow \partial X'$; however, such a homeomorphism cannot in general be obtained as a continuous extension of a Γ -equivariant quasi-isometry of X to X' .*

The author would like to thank the referee for pointing out a mistakes in the original versions of Lemmas 2.0 and 2.10 and for many other helpful suggestions.

1. BASICS ON $CAT(0)$ SPACES AND BOUNDARIES

The idea of this curvature notion is to compare a triangle in the given space with a triangle of the same size in \mathbb{E}^2 and ask that the given triangle be at least as thin as a Euclidean one of the same side lengths.

Definition. Let (X, d) be a proper complete geodesic metric space. If Δabc is a geodesic triangle in X , then we consider $\Delta \bar{a}\bar{b}\bar{c}$ in \mathbb{E}^2 , a triangle with the same side lengths, and call this a comparison triangle. Then we say X satisfies the $CAT(0)$ inequality if given Δabc in X , then for any comparison triangle and any two points p, q on Δabc , the corresponding points \bar{p}, \bar{q} on the comparison triangle satisfy

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

Some examples of $CAT(0)$ spaces are $\mathbb{E}^n, \mathbb{H}^n$, and also \mathbb{R} -trees. The universal cover of any compact Riemannian manifold of non-positive curvature is $CAT(0)$ as well [BGS]. Also, any closed convex subset of a $CAT(0)$ space is $CAT(0)$ with the induced metric.

Another way of building examples of $CAT(0)$ spaces is by taking products. If $(X_1, d_1), (X_2, d_2)$ are $CAT(0)$, then $X = X_1 \times X_2$ is $CAT(0)$ when given the metric $d = \sqrt{d_1^2 + d_2^2}$. X_1 and X_2 are closed convex subsets of X here. Our main object of study in this paper will be products.

It is worth noting that one can define a local notion of negative curvature in a similar manner. Given a geodesic metric space (X, d) and $\epsilon < 0$, one can compare geodesic triangles in X with triangles of the same size in the surface of constant curvature ϵ as above. X is called $CAT(\epsilon)$ if this is the case (examples are universal covers of compact Riemannian manifolds of negative curvature). This notion differs from the previously mentioned notion of δ -negative curvature because the first is an asymptotic notion - all things of size less than δ are ignored.

Definition. A group Γ is a $CAT(0)$ group if Γ acts geometrically on some $CAT(0)$ space (X, d) .

The classical hyperbolic and Euclidean groups are all $CAT(0)$ groups as well as fundamental groups of compact Riemannian manifolds of non-positive curvature since they act geometrically on their universal cover. Also, Coxeter groups are $CAT(0)$ groups, [Mou].

If (X, d) is a $CAT(0)$ space, then the following basic properties hold:

- (1) The distance function $d: X \times X \rightarrow \mathbb{R}$ is convex.
- (2) X has unique geodesic segments between points.
- (3) X is contractible.

For details, see [P] and [BH].

Next we define the boundary of a $CAT(0)$ space and give some basic properties of this boundary.

Let (X, d) be a proper $CAT(0)$ space. First, we can define ∂X as a set as follows:

Definition. Two geodesic rays $c, c': [0, \infty) \rightarrow X$ are said to be asymptotic if there exists a constant K such that $d(c(t), c'(t)) \leq K, \forall t > 0$ - this is an equivalence relation. The boundary of X , ∂X , is then the set of equivalence classes of geodesic rays. The union $X \cup \partial X$ will be denoted \bar{X} . The equivalence class of a ray c is denoted by $[c]$.

We state (without proof) a proposition which will guarantee that the topology we define on $X \cup \partial X$ is basepoint independent. The proof can be found in [BH, Chap 3]

Proposition 1.1. *If X is a CAT(0) space and $c: [0, \infty) \rightarrow X$ is a geodesic ray from x , then for every point $x' \in X$, there is a unique geodesic ray c' beginning at x' and is asymptotic to c .*

We define a topology on $\overline{X} = X \cup \partial X$ that induces the metric topology on X . Fix a basepoint x_0 in X and let $B(x_0, r)$ denote the ball of radius $r > 0$ about x_0 while $S(x_0, r)$ denotes the sphere of radius r about x_0 . $B(x_0, r)$ is a complete, compact, convex, subset of X and so projection onto it is well-defined. If $r' \geq r$ and $x \in S(x_0, r')$, then the image of x under projection onto $B(x_0, r)$ is the point obtained by intersecting $S(x_0, r)$ with the unique geodesic from x to x_0 . The projections $p_{r'}|_{B(x_0, r)}: B(x_0, r') \rightarrow B(x_0, r)$ for $r' \geq r$ form an inverse system. Every point is a map $c: [0, \infty) \rightarrow X$ which either becomes constant eventually or else $c(r') \neq c(r)$ for $r' \geq r$. The points of the first type have a minimum $r_0 \geq 0$ such that $c(r) = c(r_0) \forall r \geq r_0$ and so c is the unique segment from x_0 to $c(r_0)$ when restricted to $[0, r_0]$ and constant on the rest. The second type is a geodesic ray based at x_0 . Thus there is a natural bijection $\phi(x_0): \overline{X} \rightarrow \varprojlim B(x_0, r)$. We give \overline{X} the topology which makes this a homeomorphism - where $\varprojlim B(x_0, r)$ is given the inverse limit topology. This is called the *cone topology* on \overline{X} .

There are several important things to point out about this topology:

- (1) When X is proper, \overline{X} is compact and X is identified with a dense open subset of \overline{X} under inclusion.
- (2) ∂X is a closed subspace of \overline{X} .
- (3) There is a natural neighborhood basis for a point in ∂X . Let c be a geodesic ray emanating from x_0 and $r > 0$, $\epsilon > 0$. Define

$$U(c, r, \epsilon) = \{x \in \overline{X} | d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon\}$$

This consists of all points in \overline{X} such that when projected back to the sphere of radius r about x_0 , this projection is not more than ϵ away from the intersection of that sphere with c . These sets along with the metric balls about x_0 form a basis for this cone topology.

- (4) This topology is independent of basepoint - i.e. there is a natural identification between topologies when the basepoint is changed. This follows from proposition 1.1 above.
- (5) When a space X is CAT(0) AND δ -negatively curved, then the notation ∂X is unambiguous since the Gromov boundary of X as a δ -negatively curved space agrees with our above definition.
- (6) An isometry $\phi: X \rightarrow X$ extends to a homeomorphism of the boundary of X since $\phi(B(x_0, r)) = B(\phi(x_0), r)$ and conjugates projection thus giving a homeomorphism between the inverse limits.

Examples. Examples of the cone topology include the following:

- (1) $\partial \mathbb{R}^n = S^{n-1}$.
- (2) $\partial \mathbb{H}^n = S^{n-1}$
- (3) The boundary of an infinite simplicial \mathbb{R} -tree in which every vertex has

- (4) If X' is a closed, convex subset then X' is CAT(0) and $\partial X'$ embeds naturally in ∂X .
- (5) If X_1, X_2 are CAT(0) spaces, then we saw above that $X = X_1 \times X_2$ is also CAT(0). It turns out that $\partial X \cong \partial X_1 \star \partial X_2$, where \star denotes the spherical join. Indeed, since X_1, X_2 are closed, convex subsets of X , projection is well-defined so any ray c in ∂X , can be projected onto each of the factors to obtain rays in X_1 and X_2 which tell the “coordinates” of c in the join. It will be helpful to set up the following notation for these products:

$$\partial X_1 \star \partial X_2 = \frac{\partial X_1 \times [0, \frac{\pi}{2}] \times \partial X_2}{\sim}$$

where \sim denotes the equivalence relation that identifies the points (u_1, θ, u_2) and (u'_1, θ, u'_2) if and only if $[\theta = 0 \text{ and } u_2 = u'_2]$ or $[\theta = \frac{\pi}{2} \text{ and } u_1 = u'_1]$. To denote the equivalence class of the point (u_1, θ, u_2) , we use the convenient notation $u_1 \cos \theta + u_2 \sin \theta$ following [BH].

Isometries of CAT(0) spaces can be divided into three types. This classification is based on the behavior of the displacement function for an isometry.

Definition. Let γ be an isometry of the metric space X . The displacement function $d_\gamma: X \rightarrow \mathbb{R}_+$ defined by $d_\gamma(x) = d(\gamma \cdot x, x)$. The translation length of γ is the number $|\gamma| = \inf\{d_\gamma(x) : x \in X\}$. The set of points where γ attains this infimum will be denoted $\text{Min}(\gamma)$. An isometry γ is called semi-simple if $\text{Min}(\gamma)$ is non-empty.

Some basic properties about this $\text{Min}(\gamma)$ are summarized in the following proposition, see [BH, chap 3].

Proposition 1.2. Let X be a metric space and γ an isometry of X .

- (1) $\text{Min}(\gamma)$ is γ -invariant.
- (2) If α is another isometry of X , then $|\gamma| = |\alpha\gamma\alpha^{-1}|$, and $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$; in particular, if α commutes with γ , then it leaves $\text{Min}(\gamma)$ invariant.
- (3) If X is CAT(0), then the displacement function d_γ is convex: hence $\text{Min}(\gamma)$ is a closed convex subset of X .

The proofs of the first two properties are easy and the third follows directly from the fact that the distance function on X is convex. The following proposition gives the classification of isometries of X .

Definition. Let X be a metric space. An isometry γ of X is called

- (1) elliptic if γ has a fixed point - i.e $|\gamma| = 0$ and $\text{Min}(\gamma)$ is non-empty.
- (2) hyperbolic if d_γ attains a strictly positive infimum.
- (3) parabolic if d_γ does not attain its infimum, in other words if $\text{Min}(\gamma)$ is empty.

It is clear that an isometry is semi-simple if and only if it is elliptic or hyperbolic. If two isometries are conjugate in $\text{Isom}(X)$, then they are in the same class.

It is important to point out here that when a group Γ acts geometrically on a CAT(0) space X , then the elements of Γ act as semi-simple isometries because of the cocompactness of the action. Our main concern here will be the hyperbolic isometries.

There is a very important theorem about the structure of $\text{Min}(\gamma)$ when γ is a hyperbolic isometry of a $\text{CAT}(0)$ space X . The proof of this structure theorem relies on two theorems that are stated below. Both of these are proven in [BH] as well as the structure theorem. The first of these is the Flat Strip Theorem which is a generalization of a theorem that holds in the theory of nonpositively curved manifolds, see [BGS], p.17.

The Flat Strip Theorem. *Let X be a $\text{CAT}(0)$ space, and let $c: \mathbb{R} \rightarrow X$ and $c': \mathbb{R} \rightarrow X$ be geodesic lines in X . If c and c' are parallel, then the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$ is isometric to a flat strip $\mathbb{R} \times [0, D] \subseteq \mathbb{E}^2$.*

The next theorem provides a way of decomposing the set of parallel lines to a given line, into a product.

A Decomposition Theorem. *Let X be a $\text{CAT}(0)$ space and let $c: \mathbb{R} \rightarrow X$ be a geodesic line in X .*

- (1) *The union of the images of all geodesic lines $c': \mathbb{R} \rightarrow X$ parallel to c is a convex subspace X_c of X .*
- (2) *Let p be the restriction to X_c of the projection on the complete convex subspace $c(\mathbb{R})$. Let $X_c^\circ = p^{-1}(c(0))$. Then X_c° is convex (in particular, it is also $\text{CAT}(0)$) and X_c is canonically isometric to the product $X_c^\circ \times \mathbb{R}$.*

The next theorem is the structure theorem for $\text{Min}(\gamma)$ where γ is a hyperbolic isometry of a $\text{CAT}(0)$ space X . There are proofs available in [BH],[BR II].

Theorem 1.5. *Let X be a $\text{CAT}(0)$ space.*

- (1) *An isometry γ of X is hyperbolic if and only if there exists a geodesic line $c: \mathbb{R} \rightarrow X$ which is translated non-trivially by γ , namely $\gamma \cdot c(t) = c(t + a)$, for some $a > 0$. The set $c(\mathbb{R})$ is called an axis of γ . For any such axis, the number a is equal to $|\gamma|$.*
- (2) *If γ is hyperbolic, the axes of γ are all parallel to each other, and their union is $\text{Min}(\gamma)$.*
- (3) *$\text{Min}(\gamma)$ is isometric to a product $Y \times \mathbb{R}$, and the restriction of γ to $\text{Min}(\gamma)$ is of the form $(y, t) \mapsto (y, t + |\gamma|)$, where $y \in Y$, $t \in \mathbb{R}$.*
- (4) *Every isometry α which commutes with γ leaves $\text{Min}(\gamma) = Y \times \mathbb{R}$ invariant, and its restriction to $Y \times \mathbb{R}$ is of the form (γ_Y, γ_t) , where γ_Y is an isometry of Y and γ_t a translation of \mathbb{R} .*

Another application of the Flat Strip Theorem is the following generalization of the above theorem to include finitely generated abelian groups of semisimple isometries. It is called the Flat Torus Theorem.

Flat Torus Theorem. *Let Γ be a finitely generated abelian group acting properly by semi-simple isometries on a complete $\text{CAT}(0)$ space X .*

- (1) *$\text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma)$ is non-empty and splits as a product $Y \times \mathbb{E}^n$, where $n = \text{rk}_{\mathbb{Q}} \Gamma$;*
- (2) *every $\gamma \in \Gamma$ leaves $\text{Min}(\Gamma)$ invariant, respecting the product structure; γ is the identity on the first factor Y and a translation on the second factor \mathbb{E}^n ;*
- (3) *the quotient of each n -flat $\{y\} \times \mathbb{E}^n$ by this action is an n -torus.*

The proof, which appears in [BH], uses induction on n and the case $n = 1$ is exactly the Decomposition Theorem above. The above theorem tells us the following interesting fact which we will use later on:

Corollary 1.7. *If Γ is a group acting geometrically on a CAT(0) space X , and $H \cong \mathbb{Z}^n$ a subgroup of Γ , then there exists a point $x \in X$ such that the orbit $H \cdot x$ is a lattice in an isometrically embedded copy of \mathbb{E}^n (i.e. an n -flat).*

One more result which we will need in the proof of the Main Theorem is the following. A proof of this can be found in [BH].

Flat Plane Theorem. *A proper cocompact CAT(0) space X is δ -negatively curved if and only if X does not contain a subspace isometric to \mathbb{E}^2 .*

2. THE ANGULAR AND TITS METRICS

Here we develop a general technique for measuring angles between points in ∂X . We assume the reader has some knowledge of how to measure angles in a metric space, but we add the necessary definitions for completeness. Alexandrov used the method of comparison triangles to define the notion of angle between two geodesics leaving a point x in a metric space X .

Definition. *Let $c : [0, a] \rightarrow X$ and $c' : [0, a'] \rightarrow X$ be two geodesics with $c(0) = c'(0) = x_0$. Given $t \in [0, a]$, $t' \in [0, a']$, and let $\alpha_{c, c'}^{t, t'}$ denote the angle in a comparison triangle in Euclidean space at the vertex corresponding to x_0 . The (upper) angle between c, c' at x_0 is defined to be the following number:*

$$\angle_{c, c'} := \limsup_{t, t' \rightarrow 0} \alpha_{c, c'}^{t, t'}$$

Note: The lim sup is used because the limit may not always exist, but in CAT(0) spaces, the limit does exist and instead of calling it an ‘‘upper’’ angle, we call it the angle. For a proof of this, we reference [BH].

Definition. *Let X be a CAT(0) space. Given $x \in X$ and $u, v \in \partial X$, we denote by $\angle_x(u, v)$ the angle between the unique geodesic rays which issue from x and lie in the classes u and v respectively. Then we define the angle between u and v to be*

$$\angle(u, v) = \sup_{x \in X} \angle_x(u, v)$$

The following proposition gives some basic facts about angles between boundary points of a (complete) CAT(0) space. For complete proofs, see chapter 3 of [BH].

Proposition 2.1. *Let X be a CAT(0) space and let c, c' be two geodesic rays issuing from the same point $x \in X$. Let $u = c(\infty)$ and $u' = c'(\infty)$.*

- (1) *If $u \neq u'$, then there exists $t > 0$ such that $\angle_{c(t)}(u, u') > 0$; hence $\angle(u, u') > 0$.*
- (2) *The function $y \mapsto \angle_y(u, u')$ is upper semicontinuous on X .*
- (3) *The function $t \mapsto \angle_{c(t)}(u, u')$ is non-decreasing and*

$$\angle(u, u') = \lim_{t \rightarrow \infty} \angle_{c(t)}(u, u')$$

- (4) *If $\angle_x(u, u') = \angle(u, u')$, then the convex hull of $c(\mathbb{R}_+) \cup c'(\mathbb{R}_+)$ is isometric to a sector in the Euclidean plane bounded by two rays which meet at an angle $\angle(u, u')$.*

- (5) Suppose $\{x_n\}, \{y_n\}$ are sequences of points of X converging to u, u' in ∂X . Then if x_0 is a basepoint of X ,

$$\liminf_{n \rightarrow \infty} \overline{\angle}_{x_0}(x_n, y_n) \geq \angle(u, u')$$

where $\overline{\angle}_{x_0}(x_n, y_n)$ denotes the angle in a Euclidean comparison triangle.

It is easy to see that this angle satisfies the triangle inequality and so this combined with (1) from above gives us that $\angle(u, u')$ defines a metric on ∂X called the *angular metric*.

The next proposition gives a way of measuring the angle between two rays using an asymptotic formula. This formula will be useful in proving the Tits metric question for products in section 4.

Proposition 2.2. *Let x be a point in the $CAT(0)$ space X and let c, c' be two geodesic rays emanating from x with $u = c(\infty)$ and $u' = c'(\infty)$. Then*

$$\lim_{t \rightarrow \infty} \frac{d(c(t), c'(t))}{t} = [2 - 2 \cos(\angle(u, u'))]^{1/2}$$

Proof. To simplify notation, let $d(t) = \frac{d(c(t), c'(t))}{t}$ and let A denote the expression on the right hand side of the formula. First we show that the limit on the left side of the equation exists. Indeed, we see that for $t_1 < t_2$ in $[0, \infty)$,

$$d(c(t_1), c'(t_1)) \leq \frac{t_1}{t_2} d(c(t_2), c'(t_2))$$

because the distance function is convex. This tells us that $d(t)$ is monotone increasing in t . We also see that this expression is bounded by 2 independent of t since

$$d(c(t), c'(t)) \leq d(c(t), x) + d(x, c'(t)) \leq t + t = 2t$$

Thus, the limit exists.

Consider the triangle with vertices $x, c(t), c'(t)$ with angles $\alpha_t, \beta_t, \gamma_t$ respectively. Let the angles in a comparison triangle be denoted $\overline{\alpha}_t, \overline{\beta}_t, \overline{\gamma}_t$. This triangle has side lengths $t, t, d(c(t), c'(t))$. We first show $\lim_{t \rightarrow \infty} d(t) \geq A$

Applying the law of cosines to the comparison triangle, we have

$$\begin{aligned} d(c(t), c'(t))^2 &= 2t^2 - 2t^2 \cos \overline{\alpha}_t \\ &= 2t^2(1 - \cos \overline{\alpha}_t) \\ &\geq 2t^2(1 - \cos \angle_x(u, u')) \\ &= t^2(2 - 2 \cos \angle_x(u, u')) \end{aligned}$$

Where the inequality comes from the fact that $\angle_x(u, u') \leq \overline{\alpha}_t$ and the cosine function is decreasing on $[0, \pi]$. Dividing both sides by t^2 and using the fact that $\angle(u, u') \geq \angle_x(u, u')$, we get $d(t)^2 \geq A^2$. Taking the limit as $t \rightarrow \infty$ gives the desired result.

To show the reverse inequality, use the same notations as above adding the following: $d = d(c(t), c'(t))$. Apply the law of cosines to the comparison triangle to obtain the following equations:

$$\begin{aligned} d^2 &= t^2 + t^2 - 2t^2 \cos \overline{\alpha}_t \\ t^2 &= t^2 + d^2 - 2dt \cos \overline{\gamma}_t \\ t^2 &= t^2 + d^2 - 2dt \cos \overline{\beta}_t \end{aligned}$$

Substituting and combining gives $d(t) = \frac{d}{t} = \cos \overline{\gamma}_t + \cos \overline{\beta}_t$. Now we have:

$$d(t) = \cos \overline{\gamma}_t + \cos \overline{\beta}_t \leq \cos \gamma_t + \cos \beta_t$$

To finish the proof, it suffices to show the last expression above is less than $[2 - 2 \cos \alpha_t]^{\frac{1}{2}}$.

Consider a Euclidean triangle (a, b, c) with the angle at vertex a being α_t and let β, γ denote the angles at vertices b, c respectively. Let $M = \beta + \gamma = \beta_t + \gamma_t$. If we consider the function $f(x) = \cos x + \cos(M - x)$ on the interval $[0, M]$, then we must have a unique maximum obtained on this interval. This maximum, at x_0 , satisfies $\sin(M - x_0) = \sin x_0$. This will occur if we specify that the sidelengths opposite β, γ in our above triangle have length 1 because of the law of sines from Euclidean geometry - i.e $\sin \beta = \sin \gamma = \sin(M - \beta)$ in this triangle, so $\beta = x_0$. Using the fact that β maximizes this function and the law of cosines on triangle (a, b, c) , we get

$$\cos \beta_t + \cos \gamma_t \leq \cos \beta + \cos \gamma = [2 - 2 \cos \alpha_t]^{\frac{1}{2}}$$

as needed. \square

Proposition 2.3. *The expression on the left hand side of the formula in Proposition 2.2 is independent of basepoint.*

Proof. Let y be another point in X . Let r, r' denote rays beginning at y which are asymptotic to c and c' respectively. We know these exist by proposition 1.1. Now we have,

$$d(r(t), r'(t)) \leq (d(r(t), c(t)) + d(c(t), c'(t)) + d(c'(t), r'(t))) \leq 2d(x, y) + d(c(t), c'(t))$$

The first follows from the triangle inequality, the second follows from the convexity of the distance function - indeed, consider the quadrilateral formed with vertices $x, y, r(t), c(t)$, we know the side formed by $r(t)$ and $c(t)$ is no bigger than the side formed by x and y . Likewise for the pair $r'(t), c'(t)$. Dividing by t and taking limits as $t \rightarrow \infty$ gives the independence of basepoint as $\frac{2d(x, y)}{t} \rightarrow 0$ as $t \rightarrow \infty$. \square

Example. Consider the CAT(0) space $X = \mathbb{H}^2$. We know ∂X is the unit circle. For any two points on the boundary circle, there is a geodesic line in \mathbb{H}^2 joining these two points. Because the above calculation is independent of basepoint, we may as well assume the basepoint is on this line. Then one easily sees that the angle between the two points must be π . This means the angle metric on $\partial(\mathbb{H}^2)$ is discrete. This is actually true for any δ -negatively curved metric space. The formula on the right side of the above equation defines a psuedo metric on the Gromov boundary. From this, one obtains a metric analogous to the angle metric on a CAT(0) space and because any two points in the Gromov boundary can be joined by a geodesic line inside the space, we again obtain a discrete metric.

Definition. Let X be a $CAT(0)$ space with boundary ∂X . Then the Tits metric on ∂X is the length metric associated to the angular metric on ∂X and is denoted Td .

Proposition 2.4. Let X be a $CAT(0)$ space. Then the following hold:

- (1) If u, v are points of ∂X such that there is no geodesic line $c : \mathbb{R} \rightarrow X$ with $c(\infty) = u$ and $c(-\infty) = v$, then, in particular if $Td(u, v) < \pi$, then $Td(u, v) = \angle(u, v)$ and there is a geodesic segment in $(\partial X, Td)$ between u and v .
- (2) For any $u, v \in \partial X$, we have

$$\angle(u, v) = \min\{\pi, Td(u, v)\}$$

In particular the Tits metric and the angular metric induce the same topology on ∂X .

- (3) If $Td(u, v) > \pi$, then there is a geodesic $c : \mathbb{R} \rightarrow X$ with $c(\infty) = u$, and $c(-\infty) = v$.
- (4) If $c : \mathbb{R} \rightarrow X$ is a geodesic line, then $Td(c(-\infty), c(\infty)) \geq \pi$ and there is equality if and only if $c(\mathbb{R})$ bounds a flat half plane.
- (5) If the diameter of ∂X with the Tits metric is π , then every geodesic line in X bounds a flat half plane.

3. MAIN THEOREM

The contents of this section is a proof of the Main Theorem stated below:

Main Theorem. Suppose $\Gamma = G \times H$ acts geometrically on a $CAT(0)$ space X with G, H non-elementary word hyperbolic. Then the following are true:

- (1) There exist closed convex subsets X_1, X_2 of X such that G and H admit geometric actions on X_1 and X_2 respectively. Furthermore, $L(X_1) \cap L(X_2) = \emptyset$ in ∂X , where $L(X_i)$ denotes the limit set of X_i , giving $\partial X \cong \partial G \star \partial H$.
- (2) Both G and H determine quasi-convex subsets of X - i.e. for a basepoint $x_0 \in X$, the orbits $(G \times \{1_H\}) \cdot x_0$ and $(\{1_G\} \times H) \cdot x_0$ are quasi-convex subsets of X .

Before giving the proof, two known results from the theory of word hyperbolic groups will be needed. We only quote them here, but proofs of both of these can be found in [R].

Double Density. Let G be a (non-elementary) word hyperbolic group and let $g^{\pm\infty}$ denote the pair of endpoints of an axis for the infinite order element $g \in G$. The set $\{(g^\infty, g^{-\infty}) : g \in G, o(g) = \infty\}$ is dense in $\partial G \times \partial G$. In short, any geodesic line can be approximated by an axis for a hyperbolic isometry.

Lemma. Suppose M is a closed, convex, quasi-dense subset of the $CAT(0)$ space X where quasi-dense means $X \subset N_d(M)$ for some constant d . Then $\partial M = \partial X$.

Proof of the Main Theorem. The proof of the Main Theorem will be given as a series of lemmas, the first of which illustrates one of the big differences between groups of the form $G \times \mathbb{Z}^n$ and $G \times H$ where G, H are non-elementary word hyperbolic groups.

We point out that this lemma is much like the lemma from both of the known results cited above, however, our assumptions are weaker and so, although the proof is similar, especially to [R], there are adjustments to be made.

Lemma 3.1. *Let D denote a compact fundamental domain for the action of Γ on X . Let $CH(G \cdot D)$ denote the intersection of all closed, convex, $G \times \{1_H\}$ -invariant subsets of X which contain D . Then there exists a compact D' such that $G \cdot D' = CH(G \cdot D)$. In short, G acts cocompactly on its closed convex hull.*

Proof. Suppose there is no such D' . Then for each $n > 0$, $\exists x_n \in CH(G \cdot D)$, such that $d(x_n, G \cdot D) > n$. Choose a sequence of elements $(g_n, h_n) \in \Gamma$ such that $x_n \in (g_n, h_n) \cdot D$. In particular, we have a sequence of elements $\{h_n\}$ from H such that $h_n \cdot x_n \in CH(G \cdot D)$. We have the following:

$$d(D, h_n \cdot D) \geq d(h_n \cdot D, G \cdot D) = d((g_n, h_n) \cdot D, G \cdot D) \geq d(x_n, G \cdot D) - \text{diam}(D)$$

the first inequality holds since $D \subset G \cdot D$ and the last one holds because $x_n \in (g_n, h_n) \cdot D$. We know the last term tends to ∞ as n gets large. This inequality tells us that the sequence of h'_n s moves D further and further off of itself and so there must be infinitely many distinct h'_n s. Thus, without loss of generality we can assume $h_n \neq h_m$ for $n \neq m$. Next we show for any $h \in H$, the family $\{d_{h_n h h_n^{-1}}\}$ of displacement functions is uniformly bounded on $CH(G \cdot D)$. Indeed, since D is compact and the displacement functions are continuous, choose $M = \max\{d_h(x) : x \in D\}$. We know h commutes with all of G , therefore

$$d_h(x) = d(x, h \cdot x) = d((g, 1) \cdot x, (g, h) \cdot x) = d_h((g, 1) \cdot x)$$

and so d_h is constant on the orbits of G . Now let $S = \{x \in X : d_h(x) \leq M\}$. S is a closed, convex, G -invariant subspace containing $G \cdot D$, thus $CH(G \cdot D) \subset S$ and we have d_h bounded by M on $CH(G \cdot D)$. Now let $y \in D$ be arbitrary. We know

$$d_{h_n h h_n^{-1}}(h_n \cdot y) = d(h_n \cdot y, h_n h \cdot y) = d(y, h \cdot y) = d_h(y) \leq M$$

Also, since $(g_n, h_n) \cdot x_n \in D$, we have

$$\begin{aligned} d_{h_n h h_n^{-1}}(y) &= d(h_n h h_n^{-1} \cdot y, y) \\ &\leq d(y, (g_n, h_n) \cdot x_n) + d((g_n, h_n) \cdot x_n, h_n h h_n^{-1} (g_n, h_n) \cdot x_n) + \\ &\quad d(h_n h h_n^{-1} (g_n, h_n) \cdot x_n, h_n h h_n^{-1} \cdot y) \\ &\leq d((g_n, h_n) \cdot x_n, (g_n, h_n) h \cdot x_n) + 2 \cdot (\text{diam} D) \\ &= d_h(x_n) + 2 \cdot (\text{diam} D) \\ &\leq M + 2 \cdot (\text{diam} D) \end{aligned}$$

Thus we have $\{d_{h_n h h_n^{-1}}\}$ uniformly bounded on D as needed. But the action of Γ is properly discontinuous so this would imply that for a fixed $h \in H$, there are only finitely many distinct $h_n h h_n^{-1}$'s. Suppose $S = \{s_1, \dots, s_k\}$, is a finite generating set for H . For each $j = 1, \dots, k$, we can find a subsequence of the $\{h_n\}$ so that all the elements of $\{h_n s_j h_n^{-1}\}$ are equal for this subsequence. Starting with $j = 1$, we choose a subsequence that works for s_1 , now we can choose a subsequence of that subsequence which will work for s_2 , and so on until we get to k . We will then have a subsequence of the $\{h_n\}$, so that for each $j = 1, \dots, k$, the elements $\{h_n s_j h_n^{-1}\}$ are all equal. But this subsequence will work for any $h \in H$ now because S generates

all of H . So, for any $h \in H$, we have $h_n h h_n^{-1} = h_m h h_m^{-1}$ for $n \neq m$ which gives $h_m^{-1} h_n h = h h_m^{-1} h_n$ - i.e. $h_m^{-1} h_n$ is in the center of H for $n \neq m$. As the center of a non-elementary word hyperbolic group is finite, we can assume that all the $h_m^{-1} h_n$ are the same for $n \neq m$. In particular, $h_m^{-1} h_1 = h_n^{-1} h_1$ for $n \neq m$ which says $h_n = h_m$ - a contradiction since we chose them to be distinct. \square

Assertion 2 of the theorem follows easily from this lemma so we state and prove it here.

Corollary 3.2. *The orbits of G and H are quasi-convex subsets of X .*

Proof. First, since both G and H are non-elementary word hyperbolic, we can switch their roles in the above lemma to conclude that H acts cocompactly on its convex hull as well so we show it suffices to show the orbit of G is quasi-convex. Fix a basepoint $x_0 \in D'$ as above. The orbit $(G \times \{1_H\}) \cdot x_0$ is now contained in $CH(G \cdot D)$ as it is G -invariant and contains D' . Fix $(g_1, 1) \cdot x_0, (g_2, 1) \cdot x_0$ in the orbit of G . We know the segment between them is contained in $CH(G \cdot D)$ as it is convex. From the lemma, the G translates of D' cover all of $CH(G \cdot D)$. Let $K = \text{diam} D'$. For any point p on the segment $[(g_1, 1) \cdot x_0, (g_2, 1) \cdot x_0]$, we know there is a $g \in G$ so that $p \in g \cdot D'$. As $g \cdot x_0$ is also in there, we have $d(p, g \cdot x_0) < K$. Thus K is our quasi-convexity constant. \square

Let \mathcal{S} denote the set of all non-empty, minimal, closed, convex, G invariant subspaces of X . Notice that if C_1, C_2 are distinct elements of \mathcal{S} , then $C_1 \cap C_2 = \emptyset$ as the intersection of any two members of \mathcal{S} is again closed, convex, and G invariant. Also, since any decreasing sequence of elements of non-empty closed convex G -invariant subsets that are contained in $CH(G \cdot D)$ must have a non-empty intersection as each will intersect with D' . Thus we can apply Zorn's Lemma to obtain a minimal G invariant closed, convex subset in $CH(G \cdot D)$.

Remark 3.3. In [B], it is shown that any two elements of the collection \mathcal{S} are parallel - i.e. if $d = d(C_1, C_2) = \inf_{x_1 \in C_1, x_2 \in C_2} d(x_1, x_2)$, then there is a unique isometry of $C_1 \times [0, d]$ on the convex hull $CH(C_1 \cup C_2)$ of $C_1 \cup C_2$ which respects projection. Specifically, if we let $\pi_1 : X \rightarrow C_1$ denote projection onto C_1 , then this map is an isometry from C_2 onto C_1 when restricted. The proof of this uses the minimality of the elements of \mathcal{S} as well as a theorem called the Flat Quadrilateral Theorem which is proven in [BH]. This result states that if all four angles in a quadrilateral are $\geq \frac{\pi}{2}$, then, in fact, they are all $= \frac{\pi}{2}$ and the quadrilateral is actually flat. Next, it is shown that the collection of all of these parallel copies forms a closed, convex, Γ invariant subspace of X , call it X' . From the preliminary lemma, X' is quasi-dense in X , which gives $\partial X = \partial X'$. Thus for the asymptotic result, we can assume that $X = X'$.

Remark 3.4. Let C_1, C_2, C_3 be in the collection \mathcal{S} and let π_i denote projection onto $C_i, i = 1, 2, 3$. Then $\pi_1 = \pi_1 \pi_2$ for all points of C_3 that lie on a line. In [B], this works for all points of C_3 since the space X is assumed to have local extendability of geodesics. We do not assume this, so we must only use this fact for points which we know lie on a line inside C_3 .

Remark 3.5. It is convenient to set up some notation at this point. Let X_1 be an element of the collection \mathcal{S} and let $x_0 \in X_1$. Let $\pi_1 : X \rightarrow X_1$ denote projection. It is important to note here that for each $x \in X$, there is a unique C_x in the collection \mathcal{S} containing x ; this is because $\pi_1 \circ C_x \rightarrow X_1$ is an isometry for all $x \in X$.

Let g be an infinite order element in G . We know g acts as a hyperbolic isometry on X and thus on X_1 (since X_1 is convex, closed, and G -invariant) so we can find an axis for g inside X_1 . We know by the Decomposition Theorem, that $\text{Min}(g)$ (in X) is isometric to a product $Y_g \times \mathbb{R}$ where Y_g is a closed, convex subset of X .

Lemma 3.6. *Let $x \in Y_g$. Define $h * x = \pi_{Y_g}((1, h) \cdot x)$ where π_{Y_g} denotes projection onto Y_g . This action is a geometric action of H on Y_g giving $\partial Y_g \cong \partial H$.*

Proof. To show the action is geometric, we must show it is isometric, properly discontinuous, and cocompact. To show $*$ is isometric, let $x, y \in Y_g$ and $h \in H$. Now we have:

$$d(h * x, h * y) = d(\pi_{Y_g}((1, h) \cdot x), \pi_{Y_g}((1, h) \cdot y)) = d(h \cdot x, h \cdot y) = d(x, y)$$

The second equality follows because all the axes for g are parallel and hitting by h takes parallel axes to parallel axes, the same distance apart. Then since $\text{Min}(g)$ is a product, we know projection onto the Y_g factor preserves the distance between these lines. Thus the action is isometric.

Assume the action is not properly discontinuous on Y_g . Then there exists a compact $K \subset Y_g$ (containing the basepoint x_0) and a sequence $\{h_n\}$ of elements of H , such that $h_n * K \cap K \neq \emptyset$. For each n , we know there is a unique $C_{h_n \cdot x_0} \in \mathcal{S}$ such that $h_n \cdot x_0 \in C_{h_n \cdot x_0}$. We know from Lemma 3.2 that G acts cocompactly on each of the elements of \mathcal{S} , so let M be the diameter of a fundamental region for this action. Now we can choose a $g_n \in G$, such that

$$d((g_n, h_n) \cdot x_0, h_n * x_0) < M$$

Now let $\gamma_n = (g_n, h_n)$. We now have,

$$d(x_0, \gamma_n \cdot x_0) \leq d(x_0, h_n * x_0) + d(h_n * x_0, \gamma_n \cdot x_0) \leq \text{diam}K + M$$

and this contradicts the proper discontinuity of the action of Γ on X . Thus the action of H on Y_g is properly discontinuous as needed.

To show cocompactness, it suffices to show if D is a fundamental region for the action of Γ on X , then $H * D$ covers all of Y_g . Let $y \in Y_g$ and fix a basepoint $x_0 \in D$. Then there exists a $\gamma = (g, h) \in \Gamma$, such that $d(\gamma \cdot x_0, y) < \text{diam}(D)$. Since h permutes the elements of \mathcal{S} and g leaves them setwise fixed, we know $d(C_y, C_{h \cdot x_0}) < \text{diam}(D)$. Projection onto Y_g is non-increasing so

$$d(\pi_{Y_g}(h \cdot x_0), \pi_{Y_g}(y)) = d(h * x_0, y) < \text{diam}(D)$$

giving the cocompactness of the action.

Since the action of H on Y_g is geometric and H is negatively curved, we know there is a natural quasi-isometry between H and Y_g which extends to a homeomorphism of boundaries giving $\partial Y_g \cong \partial H$. \square

We have embeddings of ∂H and ∂G into ∂X . To get a homeomorphism of $\partial G * \partial H$ onto ∂X , we must show the images of these embeddings do not intersect. We use the regular metric introduced in section 2.

Lemma 3.7. *Fix an X_1 in \mathcal{S} and let Y_g be as above. Then $\partial X_1 \cap \partial Y_g = \emptyset$.*

Proof. Let $p \in \partial X_1$ and $q \in \partial Y_g$. We know X_1 is negatively curved so the angular metric is discrete on ∂X_1 - i.e. if $p \neq p' \in \partial X_1$, then $\angle(p, p') = \pi$. Also, since $\text{Min}(g) \equiv Y_g \times \mathbb{R}$, where \mathbb{R} is an axis for g , we have $\angle(q, g^\infty) = \frac{\pi}{2}$ for all $q \in \partial Y_g$. Thus we have the following:

$$\begin{aligned} \angle(p, g^\infty) &\leq \angle(p, q) + \angle(q, g^\infty) \\ \pi &\leq \frac{\pi}{2} + \angle(p, q) \\ \frac{\pi}{2} &\leq \angle(p, q) \end{aligned}$$

Thus we have $p \neq q$ as needed. \square

Let g' be a different infinite order element of G and $Y_{g'}$ be defined as above. It is clear that $Y_{g'}$ will admit a geometric H action as well since H commutes with all elements of G . We need to show that no matter what $g \in G$ we choose, we end up with the same copy of ∂H in ∂X . Before doing so, we make an observation about the structure of Y_g in this setting.

Lemma 3.8. *The intersection of Y_g with X_1 is a closed, bounded, convex subset of X_1 .*

Proof. $\text{Min}(g) = Y_g \times \mathbb{R}$ where Y_g parametrizes the space of all axes for g in X . Observe that $Y_g \cap X_1$ parametrizes the set of those axes in X_1 which is precisely the set of axes for the action of g on X_1 . Thus we can apply the Decomposition Theorem to the action of g on X_1 to see that $\text{Min}(g) \cap X_1 = Z_g \times \mathbb{R}$ which gives $Y_g \cap X_1 = Z_g$.

It is left to see that Z_g is bounded. This follows directly from the Flat Plane Theorem stated in section 1. Indeed, if Z_g were unbounded, then it would contain a geodesic ray which would force X_1 to contain a totally geodesic half-plane contradicting the fact that X_1 is δ -negatively curved. \square

Definition. *Two closed, convex subsets of X are called parallel if the Hausdorff distance between them is finite.*

Lemma 3.9. *Let $g, g' \in G$ be infinite order elements. Then Y_g and $Y_{g'}$ are closed, convex, subsets of X which are parallel.*

Proof. We know $Y_g, Y_{g'}$ are closed and convex by the Decomposition Theorem. Their boundaries are homeomorphic because they are δ -negatively curved and quasi-isometric (i.e. they both admit geometric H -actions). Thus it suffices to show the Hausdorff distance between them is finite to see that they determine the same subset of ∂X .

By Lemma 3.8 we have Z_g and $Z_{g'}$ closed, bounded, convex subsets of X_1 . Let $D = d(Z_g, Z_{g'}) = \min\{d(x, y) : x \in Z_g, y \in Z_{g'}\}$.

Let $x \in Y_g$ and C_x denote the unique element of \mathcal{S} containing x . Recall that $\pi_1|_{C_x} : C_x \rightarrow X_1$ is an isometry and note that $\pi_1(x) \in Z_g$. We know there exists a point $y \in Z_{g'}$ with $d(y, \pi_1(x)) \leq D$. Then $x' = \pi_1^{-1}(y)$ is a single point of C_x with $d(x, x') = d(\pi_1(x), y) \leq D$. It is now clear that Y_g and $Y_{g'}$ have finite Hausdorff distance. \square

It is left to describe where the line segments between points of ∂G and ∂H go. First notice the embedding of ∂H we are using is NOT coming from the induced action of H on X whereas the embedding of ∂G is

Lemma 3.10. *Fix a $C \in \mathcal{S}$, an infinite order $g \in G$, and a basepoint $p \in \text{Min}(g) \cap C$. Then $\text{Min}(g) = Y_g \times \mathbb{R}$ with $\partial Y_g \cong \partial H$ and $\partial C \cong \partial G$. For any $x \in \partial Y_g$ and any $y \in \partial C$, $\angle(x, y) = \frac{\pi}{2}$.*

Proof. First we consider the rational points of ∂G and ∂H . Suppose $g^\infty \in \partial G$ and $h^\infty \in \partial H$. Since g, h commute, the Flat Torus Theorem, implies there is a 2-flat in $\text{Min}(g) \cap \text{Min}(h)$ left invariant by the $\mathbb{Z} \times \mathbb{Z}$ generated by g, h where the orbit of a basepoint $x_0 \in \text{Min}(g) \cap \text{Min}(h)$ under the $\mathbb{Z} \times \mathbb{Z}$ generating a lattice inside the plane. If we choose this x_0 in C , then the g -axis, A_g , of this plane is inside C , but the h -axis is not necessarily inside Y_g . However, when we project the h -axis to Y_g we obtain an axis A_h^* , for the $*$ action of h on Y_g . The endpoint of A_h^* inside ∂Y_g corresponds to the point h^∞ under the natural homeomorphism between ∂Y_g and ∂H coming from the $*$ action. $A_g \times A_h^*$ lies in the product $\text{Min}(g) = Y_g \times \mathbb{R}$ and the convex hull of these is the 2-flat above. Now, $h^\infty \in \partial Y_g$ and $g^\infty \in \partial \mathbb{R}$ implies $\angle(g^\infty, h^\infty) = \frac{\pi}{2}$.

Consider arbitrary points $x \in \partial C$, $y \in \partial Y_g$. C, Y_g are negatively curved so pairs of rational points are dense in $\partial C \times \partial C$ and $\partial Y_g \times \partial Y_g$ by the Double Density result. First observe that we already have $\angle(g^\infty, y) = \frac{\pi}{2}$ for any $g^\infty \in \partial C$ and any $y \in \partial Y_g$. Indeed, this follows simply because $\text{Min}(g)$ splits as a product - i.e. the argument for an arbitrary $y \in \partial Y_g$ is exactly the same as it is for an $h^\infty \in \partial Y_g$. This means we will only have to do our approximating in the G -factor.

Let $x \in \partial C$ be arbitrary. We have an axis for g^∞ and our basepoint p lies on this axis. We can choose p so that when we consider the triangle with vertices $x, g^\infty, g^{-\infty}$, p lies within 2δ of the side determined by $x, g^{-\infty}$, where δ is the negative curvature constant for C .

To build a sequence of rational points in ∂C converging to x , we do the following: to obtain the n th element of the sequence, g_n , use the Double Density result on the pair $g^{-\infty}, x$ to obtain an axis for g_n with $(g_n^\infty \cdot x) > n$ and $(g_n^{-\infty} \cdot g^{-\infty}) > n$ where (\cdot) denotes the Gromov inner product. This will force the axis for g_n to pass near the point p for large enough n . In fact, $g^\infty, g_n^\infty, g^{-\infty}$, and $g_n^{-\infty}$ form a quadrilateral with vertices in ∂C so we can find a point q_n on the axis for g_n with

$$d(p, q_n) < 4\delta$$

The proof of this uses the thin triangle condition twice and is a standard argument in this context - intuitively, quadrilaterals are also "thin" in a negatively curved space so p lies within 4δ of a point on one of the other three sides. Because of our choice of p, q_n will have to lie on the axis for g_n for large enough n .

To measure $\angle(x, y)$, we first measure $\angle(g_n^\infty, y)$. Since q_n lies on an axis for g_n , we have $q_n \in \text{Min}(g_n)$. Consider $Y_{g_n} = \pi^{-1}(q_n)$ where π denotes projection onto the axis for g_n . Just as above, the ray c_n in Y_{g_n} beginning at q_n and ending at y (i.e. the point in ∂Y_{g_n} corresponding to y under the homeomorphism onto ∂Y_g) along with the ray d_n from q_n to g_n^∞ will form a Euclidean quadrant and so

$$\angle(g_n^\infty, y) = \frac{\pi}{2}$$

For all $n, q_n \in B(4\delta, p)$, a compact set, so there is a $q \in B(4\delta, p)$ such that $q_n \rightarrow q$. Since $g_n^\infty \rightarrow x$, the rays d_n converge to a ray, d , beginning at q and ending at x . All of the rays c_n have their endpoint at $y \in \partial X$ and so converge to a ray, c , from

Pick any $\epsilon > 0$. We know the geodesic triangle with vertices $q_n, c_n(\epsilon), d_n(\epsilon)$ satisfies the pythagorean identity (because they are in a Euclidean quadrant). Since geodesics vary continuously with respect to endpoints, we know the segments $[q_n, d_n(\epsilon)]$ converge to the segment $[q, d(\epsilon)]$, likewise $[q_n, c_n(\epsilon)]$ converge to $[q, c(\epsilon)]$. Thus the geodesic triangle with vertices $q, c(\epsilon), d(\epsilon)$ also satisfies the pythagorean identity. This is true if and only if the angle at q is exactly $\frac{\pi}{2}$. \square

Lemma 3.11. *There is a homeomorphism of $\partial G \star \partial H$ onto ∂X .*

Proof. Fix C, Y_g as above. We have shown how to map $\partial G, \partial H$ onto $\partial C, \partial Y_g$ respectively. We have also shown that for $x \in \partial C$ and $y \in \partial Y_g$, $\angle(x, y) = \frac{\pi}{2}$. In fact, we've shown these are the endpoints of rays which bound a Euclidean quadrant. Thus we can map the line segment between $x \in \partial G$ and $y \in \partial H$ (i.e. the points corresponding to x, y under the homeomorphisms between boundaries) to the line segment this quadrant determines in ∂X . It is left to show this map is a homeomorphism where we are using the cone topology on ∂X .

That this map is one-to-one and continuous is clear, thus it is left to show that map is onto. Let $z \in \partial X$. If $z \in \partial C$ or $z \in \partial Y_g$, then we are done. Otherwise, let r denote the ray beginning at p and ending at z and project r onto C under the projection map π_C . We claim that the image of this projection is unbounded and thus determines a ray in C .

Indeed, suppose the image of r under π_C lies in some K ball around the point p . Since z not in ∂C , we know for each $t > 0$, $r(t)$ lies in a unique parallel copy of C denoted by C_t and for $s \neq t$, $C_t \neq C_s$ by the convexity and minimality of each copy of C . Let p_t denote the image of p under the projection of C onto C_t - notice each of these is in Y_g (i.e. - since A_g entirely contained in C , p closest point projection of p_t in C implies p is closest point projection on A_g). We know $p, p_t, r(t), \pi_C(r(t))$ are the vertices of a flat quadrilateral, thus the segment $[p, r(t)]$ is the diagonal of this quadrilateral. Since $d(p, r(t)) \rightarrow \infty$ as $t \rightarrow \infty$ we see that $d(p, p_t) \rightarrow \infty$ as well. Then $[p, p_t]$ would determine a ray in Y_g with endpoint in ∂Y_g . But for all $t > 0$ we now have

$$d(r(t), p_t) = d(p, \pi_C(r(t))) < K$$

which implies that z , the endpoint of r lies in ∂Y_g which is contrary to our assumption.

A symmetric argument shows that the image of r projected onto Y_g is also unbounded. Then $\pi_C(im(r))$ is a ray in C beginning at p and so has an endpoint we denote by $\pi_C(z) \in \partial C$, likewise we have $\pi_{Y_g}(z) \in \partial Y_g$. We know from the previous lemma that $\angle(\pi_C(z), \pi_{Y_g}(z)) = \frac{\pi}{2}$ and so the two points determine a Euclidean quadrant with origin p . The ray r lies entirely in this quadrant and so the endpoint, z of r , lies on the segment that the quadrant determines in ∂X showing that the map is onto. \square

This completes the proof of the Main Theorem. \square

4. The Tits Metric Question

In section 2 we discussed the angular and Tits metrics on the boundary of a $CAT(0)$ space X . In this section we provide a proof of:

Theorem 4.1. *Suppose $X = X_1 \times X_2$ is a $CAT(0)$ space. Then $(\partial X, \angle)$ is isometric to the spherical join of $(\partial Y_1, \angle)$ and $(\partial Y_2, \angle)$, explicitly for any two points*

$u = u_1 \cos \theta + u_2 \sin \theta$ and $u' = u'_1 \cos \theta' + u'_2 \sin \theta'$ in ∂X , we have:

$$\cos \angle(u, u') = \cos \theta \cos \theta' \cos \angle(u_1, u'_1) + \sin \theta \sin \theta' \cos \angle(u_2, u'_2)$$

Proof. Choose basepoints $x_i \in X_i$ for $i = 1, 2$. Let c_i, c'_i denote geodesic rays issuing from x_i representing the points $u_i, u'_i \in \partial X_i$, $i = 1, 2$ respectively. Then the rays c, c' defined by $c(t) = (c_1(t \cos \theta), c_2(t \sin \theta))$ and $c'(t) = (c'_1(t \cos \theta'), c'_2(t \sin \theta'))$ are geodesic rays in X with $c(\infty) = u$ and $c'(\infty) = u'$. Now we simply apply the asymptotic formula for calculating angles (Prop 2.2) to obtain:

$$\begin{aligned} 2 - 2 \cos \angle(u, u') &= \lim_{t \rightarrow \infty} (d(c(t), c'(t))/t)^2 \\ &= \lim_{t \rightarrow \infty} (d(c_1(t \cos \theta), c'_1(t \cos \theta'))/t)^2 + \\ &\quad \lim_{t \rightarrow \infty} (d(c_2(t \sin \theta), c'_2(t \sin \theta'))/t)^2 \\ &= \cos^2 \theta + \cos^2 \theta' - 2 \cos \theta \cos \theta' \cos \angle(u_1, u'_1) + \sin^2 \theta + \\ &\quad \sin^2 \theta' - 2 \sin \theta \sin \theta' \cos \angle(u_2, u'_2) \\ &= 2 - 2 \cos \theta \cos \theta' \cos \angle(u_1, u'_1) - 2 \sin \theta \sin \theta' \cos \angle(u_2, u'_2), \end{aligned}$$

which gives the required result. \square

Theorem 4.2. *Suppose Γ is a group of the form $G \times \mathbb{Z}^n$ OR $G \times H$ where G, H denote non-elementary word hyperbolic groups. Then if Γ acts geometrically on two CAT(0) spaces X, X' , then their boundaries are isometric in the angular and Tits metrics.*

Proof. By the theorem quoted in the introduction, we know that any space X which admits a geometric group action by a $\Gamma = G \times \mathbb{Z}^n$ must have a closed, convex, Γ -invariant subset of the form $Y \times \mathbb{R}^n$ with Y δ -negatively curved, thus $\partial X \cong \partial Y \star \partial \mathbb{R}^n$. Y δ -negatively curved implies $(\partial Y, \angle)$ is discrete and we saw in the examples that $(\partial \mathbb{R}^n, \angle)$ is homeomorphic to the standard S^{n-1} . So if Γ acts geometrically on X, X' , we have: $(\partial X, \angle)$ and $(\partial X', \angle)$ both isometric to the spherical join of a discrete metric and the standard metric on the $(n-1)$ sphere and thus isometric to each other by Theorem 4.1.

By the Main Theorem, we know that any space X which admits a geometric group action by a $\Gamma = G \times H$ must have closed, convex, subsets X_1, X_2 , both negatively curved with $\partial X \cong \partial X_1 \star \partial X_2$. Thus if Γ acts on X, X' , we have $(\partial X, \angle)$ and $(\partial X', \angle)$ both isometric to the spherical join of two discrete metric spaces by Theorem 4.1.

Now observe if X admits an action by $\Gamma = G \times H$ or $G \times \mathbb{Z}^n$, then for any two $u, v \in \partial X$, we have $\angle(u, v) \leq \pi$. Thus by Proposition 2.4, we see that $Td(u, v) = \angle(u, v)$ showing the isometries above work for the Tits metric as well here. \square

5. Density of Periodic Flats

The purpose of this section is to prove that the periodic flats are dense among all flats in CAT(0) spaces which admit a geometric action by a group of the form $\Gamma = G \times H$. There are only 1-flats and 2-flats in these spaces and we will consider these separately.

Lemma 5.1. *Suppose X is a $CAT(0)$ space with closed, convex, subsets X_1, X_2 such that the Tits metric on ∂X is isometric to the join of the Tits metrics on X_1, X_2 . Then if $u, u' \in \partial X$ are the endpoints of a geodesic line in X and $u = u_1 \cos \theta + u_2 \sin \theta$ for $\theta \neq 0, \frac{\pi}{2}$, then $u' = u'_1 \cos \theta + u'_2 \sin \theta$.*

Proof. Since we are assuming $\theta \neq 0, \frac{\pi}{2}$, we know the line is not contained in one of the factors - i.e. the endpoint u is not in one of the factors of the join, so neither is u' . Since u, u' are the endpoints of a geodesic line in X which bounds a half plane, we know $Td(u, u') = \pi$ by Proposition 2.4, so there is a segment in ∂X joining these points.

It is clear that to obtain a segment of length π in the Tits metric, start at u , move along a segment of length $\frac{\pi}{2} - \theta$ to obtain the point $u_2 \in \partial X_2$, then along a segment of length $\frac{\pi}{2}$ to the point $u'_1 \in \partial X_1$, then finally a segment of length θ to arrive at the point u' . This shows that u' has the desired form. \square

Theorem 5.2. *Suppose $\Gamma = G \times H$ where both G and H are non-elementary, word hyperbolic groups. Suppose X is a $CAT(0)$ space on which Γ acts geometrically, then the periodic 1-flats are dense among all 1-flats in X .*

Proof. From the Main Theorem, we know $\partial X = \partial X_1 \star \partial X_2$ where $\partial X_1 \cong \partial G$ and $\partial X_2 \cong \partial H$.

Let $r : \mathbb{R} \rightarrow X$ be an isometry so that the image of r in X is a 1-flat. Then endpoints of r in ∂X are denoted $r(\pm\infty)$. We have $r(\infty) = u_1 \cos \theta + u_2 \sin \theta$ for some $\theta \in [0, \frac{\pi}{2}]$. If $\theta = 0$, then $r(\infty) = u_1 \in \partial X_1$ and if $\theta = \frac{\pi}{2}$, then $r(\infty) = u_2 \in \partial X_2$. Since both X_1, X_2 are closed, convex, this would imply the entire line r is in one of the factors. In these cases, we can apply the Double Density result since both factors are δ -negatively curved.

Thus from now on, we assume $\theta \in (0, \frac{\pi}{2})$. $r(\infty) = u_1 \cos \theta + u_2 \sin \theta$ means $u_i \in \partial X_i, i = 1, 2$ and (see Lemma 5.1 above)

$$\begin{aligned} \angle(r(\infty), u_1) &= \theta \\ \angle(r(\infty), u_2) &= \frac{\pi}{2} - \theta \end{aligned}$$

Since r is a geodesic line in X joining $r(\pm\infty)$, Lemma 5.1 implies $r(-\infty) = u'_1 \cos \theta + u'_2 \sin \theta$. Since $u_1, u'_1 \in \partial X_1$ and X_1 negatively curved, there is a geodesic line in X_1 joining these two points - likewise, for $u_2, u'_2 \in \partial X_2$. Next apply the Double Density result to the pairs $u_i, u'_i, i = 1, 2$ to obtain $g \in G, h \in H$ both of infinite order with

$$\begin{aligned} (g^\infty, g^{-\infty}) &\quad \text{"close to"} \quad (u_1, u'_1) \\ (h^\infty, h^{-\infty}) &\quad \text{"close to"} \quad (u_2, u'_2) \end{aligned}$$

where "close" means in $\partial X_i \times \partial X_i$ for $i = 1, 2$ where each carries the cone topology. Note here that the axis for h is not from the action of Γ on X , but from the action of H on X_2 .

Since g, h commute, they generate a copy of $\mathbb{Z} \oplus \mathbb{Z}$ in Γ , thus by the Flat Torus Theorem, there is an isometric embedding of \mathbb{R}^2 inside X on which this subgroup acts (under the induced action from Γ on X) by a torus action giving a quasi-dense lattice of orbit points in this plane. The boundary of this plane is a circle which is embedded in ∂X . The rational slope lines in this plane are the axes of elements of the form $(g, 1_H)^m \cdot (1_G, h)^n \in \Gamma$. Simply choose a line of rational slope q , with q close to θ to obtain an axis with the desired properties. \square

Theorem 5.3. *Suppose Γ and X as in the previous theorem. Then the periodic 2-flats are dense among all 2-flats in X .*

Proof. Since $\partial X = \partial X_1 \star \partial X_2$ with $X_i, i = 1, 2$ negatively curved, we know from Theorem 4.2 that the Tits metric on ∂X is the join of two discrete metrics.

If F is a 2-flat in X , then we know $\partial F = S^1$ embeds in ∂X . Using an analysis similar to that done above, we observe there are only two ways to embed an S^1 in the join of two discrete metrics. Thus we can reduce to the case where the image of the y -axis has endpoints $y(\pm\infty) \in \partial X_2$, the image of the x -axis has endpoints $x(\pm\infty) \in \partial X_1$, and $x_0 \in X$ is the image of the origin.

Because the Double Density result applies to both factors we can find elements $g \in G$ and $h \in H$, both of infinite order whose axes approximate the images of the x and y axes respectively. Just as in the proof of Lemma 3.10, we can choose these so that the axes both pass 4δ close to x_0 . The $\mathbb{Z} \oplus \mathbb{Z}$ subgroup generated by g and h determines a periodic 2-flat which will be close to F . Now build a sequence just as in Lemma 3.10 to get Hausdorff convergence. \square .

REFERENCES

- [A] A.D. Alexandrov, *A theorem on triangles in a metric space and some applications*, Trudy Mat. Inst. Steks. **38** (1951), 5–23. (Russian)
- [BGS] W. Ballman, M. Gromov, V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser, 1985.
- [B] C. Baribaud, *Un théorème de décomposition dans un espace géodésique vérifiant l'inégalité CAT(0)*, diplôme work (1992), l'Université de Genève. (French)
- [BR I] P. Bowers and K. Ruane, *Fixed points in boundaries of negatively curved groups*, to appear in Trans. of AMS.
- [BR II] P. Bowers and K. Ruane, *Boundaries of non-positively curved groups of the form $G \times \mathbb{Z}^n$* , to appear in Glasgow Math. Jour..
- [BH] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, unpublished.
- [GW] D. Gromoll and J. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of non-positive curvature*, Bull. A.M.S. **77** (1971), 545–552.
- [Gr] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol. 2, Cambridge Univ. Press, 1993, pp. 182.
- [Mou] G. Moussong, *Hyperbolic Coxeter Groups*, Ph.D. thesis (1988), The Ohio State University.
- [P] F. Paulin, *Constructions of hyperbolic groups via hyperbolizations of polyhedra*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky, eds.), World Scientific, 1991, pp. 313–328.
- [R] K. Ruane, *Boundaries of Groups*, Ph.D. thesis (1996), Florida State University.