

# LOCAL CONNECTIVITY OF RIGHT-ANGLED COXETER GROUP BOUNDARIES

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ABSTRACT. We provide conditions on the defining graph of a right-angled Coxeter group presentation that guarantees the boundary of any  $CAT(0)$  space on which the group acts geometrically will be locally connected.

## 0. Introduction

In [BM], the authors ask whether all one-ended word hyperbolic groups have locally connected boundary. In that paper, they relate the existence of global cut points in the boundary to local connectivity of the boundary. In [Bo], Bowditch gives a correspondence between local cut points in the boundary and splittings of the group over one-ended subgroups. In [S], Swarup uses the work of Bowditch and others to prove that the boundary of such a group cannot contain a global cut point, thus proving these boundaries are locally connected.

The situation is quite different in the setting of  $CAT(0)$  groups. If  $G$  is a one-ended group acting geometrically on a  $CAT(0)$  space  $X$ , then  $\partial X$  can indeed be non-locally connected. For example, consider the group  $G = F_2 \times \mathbb{Z}$  where  $F_2$  denotes the non-abelian free group of rank 2, acting on  $X = T \times \mathbb{R}$  where  $T$  is the tree of valence 4 (or the Cayley graph of  $F_2$  with the standard generating set). It is easy to see that  $\partial X = \Sigma(C)$  (where  $\Sigma$  denotes the unreduced suspension), for  $C$  a Cantor set and thus  $\partial X$  is not locally connected. Another way of viewing this group is as an amalgamated product of two copies of  $\mathbb{Z} \oplus \mathbb{Z}$  over a  $\mathbb{Z}$  subgroup with presentation  $\langle a, b, c, d \mid ab = ba, cd = dc, b = c \rangle$ . This group naturally acts on a  $CAT(0)$  space  $X$  which is a union of planes and strips glued according to the amalgamation - in fact,  $X$  is simply the universal cover of the presentation two complex where each square is given the metric of a unit square in the plane. In [MR], we prove it is easy to construct these types of examples using amalgamated products that are “geometric” in some sense. In particular, a more general version of the main theorem of that paper is given here as Theorem 3.3 and the above example clearly satisfies the hypotheses.

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**Theorem 3.3** *Suppose  $A$ ,  $B$  and  $C$  are finitely generated groups and  $G = A *_C B$  acts geometrically on a  $CAT(0)$  space  $X$ . If the following conditions are satisfied, then  $\partial X$  is not locally connected:*

- (1)  $[A : C] \geq 2$ ,  $[B : C] \geq 3$ .
- (2) *There exists  $s \in G - C$  with  $s^n \notin C$  for all  $n \neq 0$  and  $sCs^{-1} \subset C$ .*
- (3)  $Cx_0$  is quasi-convex in  $X$  for a basepoint  $x_0$ .

A general question arising naturally here is whether there is a converse to this theorem. In other words, is there a relationship between the topology of the boundary and certain “geometric” splittings of the groups? It is important to note here that the boundary of a  $CAT(0)$  group is not a well-defined object as is the case for a word hyperbolic group. Indeed, in [CK] there is an example of a group  $G$  acting geometrically on two  $CAT(0)$  spaces with non-homeomorphic boundaries. Nonetheless, any possible boundary for this group will be non-locally connected because this group admits a “geometric” splitting as in the above theorem.

Obtaining a general converse for  $CAT(0)$  groups is a very difficult question and as of yet, there is no machinery in place to prove such a result, thus it is worthwhile to test whether such a result holds for a class of  $CAT(0)$  groups that have some structure to work with. In this paper, we consider this question for right-angled Coxeter groups. Right-angled Coxeter groups were shown to be  $CAT(0)$  groups in [G] while general Coxeter groups were shown to be  $CAT(0)$  in [Mo].

In [MR], the authors give a consequence of Theorem 3.3 above for testing whether the  $CAT(0)$  Coxeter complex for a general Coxeter group has non-locally connected boundary using a presentation graph for the group. Suppose  $G$  is a Coxeter group and  $S$  is a generating set for  $G$  which gives rise to a Coxeter presentation  $P = P(G, S)$  for  $G$ . The presentation can be encoded in a graph  $\Gamma = \Gamma(G, S)$  called the *presentation graph*, whose vertex set is  $S$  and edge set comes from the relations of  $P$  (see Section 1 for details). Loosely, the consequence says (it is stated below as Theorem 3.2), if certain graph-theoretic conditions hold for  $\Gamma$ , then the boundary of the  $CAT(0)$  Coxeter complex contains points of non-local connectivity. The graph-theoretic conditions are the weakest possible conditions to guarantee that the Coxeter group splits as an amalgamated product as in Theorem 3.3. where the factors in the splitting are subgroups generated by subsets of the generating set. It is Theorem 3.2 for which we are trying to obtain a converse in the right-angled case. Our first step is to show that if the graph-theoretic conditions hold, then the boundary of any  $CAT(0)$  space on which the group acts will have non-locally connected boundary, not just the boundary of the  $CAT(0)$  Coxeter complex. This is the contents of Theorem 3.2. An improved version of Theorem 3.3 is necessary to obtain this so we provide proofs of both theorems here in Section 3. These results can be omitted if one is only interested in the main theorem.

The statement of the main theorem is given below. To understand the statement of the main theorem, we briefly explain notation and terminology here, but give formal definitions in the paper.

If  $(G, S)$  is a Coxeter system with finite generating set, then we will denote the presentation graph by  $\Gamma(G, S) = \Gamma(S) = \Gamma$  if the  $G$  and  $S$  are clear. We denote the Cayley graph by  $\Lambda(G, S) = \Lambda$ . We say a subset  $C$  of  $S$  is a *product separator* of  $\Gamma$  if  $C = A \cup B$  with  $A \cap B = \emptyset$ ,  $\langle C \rangle = \langle A \cup B \rangle = \langle A \rangle \oplus \langle B \rangle$  and  $C$  separates the graph  $\Gamma$ . A *virtual factor separator* (VFS) is a triple  $(C, C_1, K)$  where  $C$  separates  $\Gamma$ ,  $\langle C_1 \rangle$  has finite index in  $\langle C \rangle$ ,  $\langle K \rangle$  is infinite and all the letters of  $K$  commute with the letters of  $C_1$ . We may reduce the general problem of interest to the case when  $G$  does not *visually* split as a direct product. This means the graph  $\Gamma$  cannot be decomposed into two non-empty disjoint sets where each vertex of one set is joined to every vertex of the other set and vice versa. Thus we can assume not all the letters of  $S - C$  commute with all the letters in  $C$ . And finally, a *suspended separator* is a special type of VFS where  $S = C \cup \{s, t\}$  and  $\Gamma$  (as a graph) is the suspension of  $C$  with  $s$  and  $t$  as the suspension points. In this case,  $(C, C, \{s, t\})$  forms a VFS.

**Main Theorem** *Suppose  $(G, S)$  is a right-angled Coxeter system,  $G$  is one-ended,  $\Gamma(S)$  contains no product separator and no VFS. Also assume that  $G$  does not visually split as a non-trivial direct product with infinite factors. Then  $G$  has locally connected boundary.*

Note, if  $G$  has more than one end, then it is trivial to determine if  $G$  has locally connected boundary or not so it is reasonable to assume  $G$  is one-ended. If  $G$  splits as a visual direct product, then  $G$  has non-locally connected boundary iff one of the factor groups does, so for the entire paper, we assume the group  $G$  does not visually split as a direct product unless we specifically indicate that it does. We now state Theorem 3.2 which is a consequence of Theorem 3.3 above. This result sheds light on why the other hypotheses are necessary in the Main Theorem.

**Theorem 3.2** *Suppose  $(G, S)$  is a Coxeter group with presentation graph  $\Gamma$ .*

- (1) *If  $\Gamma$  has a suspended separator  $C$ , then  $G$  has locally connected boundary if and only if  $\langle C \rangle$  has locally connected boundary.*
- (2) *If  $\Gamma$  has a VFS  $(C, C_1, K)$  and  $C$  is not a suspended separator, then  $G$  has non-locally connected boundary.*

To see the extent to which our Main Theorem gives a converse to Theorem 3.3, we point out the following combination of the Main Theorem and Theorem 3.2.

**Corollary** *Suppose  $(G, S)$  is a right-angled Coxeter system,  $G$  is one-ended and  $G$  does not visually split as a non-trivial direct product. If the presentation graph  $\Gamma$  contains a VFS, then  $G$  has non-locally connected boundary. If*

*G contains no product separator and no VFS, then G has locally connected boundary.*

In Sections 4 and 5, the main technical tools for the proof are developed. From this point on, all our groups are right-angled since our methods do not work in general Coxeter groups. Section four contains two important lemmas which allow us to approximate “close”  $CAT(0)$  geodesic rays by Cayley graph geodesic rays that have a common initial subsegment (lemmas 4.2 and 4.4). In general  $CAT(0)$  group theory, one cannot hope to use Cayley graph geometry to study the geometry of geodesics in the  $CAT(0)$  space since the natural quasi-isometry between them is badly behaved. In this paper, we prove that if the group is a right-angled Coxeter group, we can gain enough control over this quasi-isometry to use the intuition coming from the Cayley graph geometry inside the  $CAT(0)$  space. This is truly the main technical difficulty in the paper.

Section 5 contains the construction of a *filter* for a pair of Cayley graph geodesics needed to prove the main theorem. In doing so, we build a collection of graphs in the plane whose edges are labeled by elements of the generating set  $S$ . There is an obvious map from any such graph into the Cayley graph  $\Lambda$ . A final version of these graphs will be 1-ended and the natural map from this graph into  $\Lambda$  will be *proper*. If the two geodesics we start with have a long common initial piece, we will show that this filter maps over to  $X$  under the natural quasi-isometry in a controlled manner.

In Section 6, we give the proof of the main theorem. Our goal is to prove local connectivity of the boundary so we start by considering two “close” geodesic rays (equivalently, two close boundary points) in the  $CAT(0)$  space. From these, we obtain two approximating Cayley graph geodesic which share a long common subpiece. Using Section 5, we build a filter between these two rays which has a nice property. In particular, the graph-theoretic hypotheses allow us to conclude there is a global bound on the length of a *factor path* in the filter (see lemma 5.10). A factor path is an edge path in the Cayley graph that lies in a product subgroup (see definition 5.2) - these are exactly the paths that behave badly under quasi-isometry. Carrying this information back to  $X$  via the natural quasi-isometry allows us to construct a “small” connected set in the boundary of the space containing the two points and hence show the  $CAT(0)$  space has locally connected boundary.

## 1. COXETER GROUP PRELIMINARIES

In this section, we prove several technical facts about Coxeter groups. Many of these facts can be found scattered about in the vast literature on Coxeter groups, however we state them here with our notation and with our particular viewpoint in mind. Even though Theorem 6.1 only holds for right-angled Coxeter groups, we prove the results in this section assuming  $G$  is a general Coxeter group. We will need the more general results in section 3 of this paper.

There are two types of results in this section. The results up through lemma 1.9 give geometric information about the Cayley graph of a Coxeter group. The most important lemmas for understanding the proof of the main theorem are theorem 1.1 and lemmas 1.4, 1.5, 1.7. We need these geometric results in sections 4 and 5 where we must compare the geometry of the  $CAT(0)$  space  $X$  with the combinatorial properties of the Cayley graph  $\Lambda$ . The remaining results culminate with lemma 1.16. This final lemma says if  $H$  is a *special* subgroup of  $G$  of finite index, then  $G$  splits as a direct product in a way that you can see in the presentation graph of  $G$ . This lemma is used to prove lemma 5.3 which is necessary to begin building a filter for a pair of geodesic rays.

A *Coxeter group* is a group  $G$  with generating set  $S = \{s_1, \dots, s_n\}$  having a *Coxeter presentation* of the form  $P(G, S) = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$  for  $1 \leq i, j \leq n$  with  $m_{ij} = 1$  if and only if  $i = j$ , otherwise  $2 \leq m_{ij} \leq \infty$ . If  $m_{ij} = \infty$ , then  $s_i, s_j$  have no relation. The pair  $(G, S)$  is called a *Coxeter System*. If  $m_{ij} = 2$  or  $\infty$  for all relations in the presentation,  $G$  is called *right-angled*.

The *presentation graph* of  $(G, S)$  is the graph  $\Gamma(G, S)$  with vertex set  $S$  and an edge labeled  $m$  between vertices  $s$  and  $t$  if  $(st)^m = 1$  is a relation of  $P$ . For right-angled groups, we omit the  $m$  labels, thus two generators commute if and only if there is an edge between them in the presentation graph.

Assume  $(G, S)$  is a Coxeter system with  $S$  a finite set. Let  $\Lambda(G, S) = \Lambda$  be the Cayley graph of  $G$  with respect to the generating set  $S$ . We assume each edge of  $\Lambda$  is labeled by an element of  $S$ . Each  $s \in S$  is an element of order two and has a fixed point set in  $\Lambda$  denoted by  $Fix(s)$ . This set consists of the midpoints of edges labeled by  $s$  with each endpoint in the subgroup of  $G$  generated by the letters of  $S$  that are adjacent to  $s$  in  $\Gamma(G, S)$ .

An *edge path* in  $\Lambda$  can be denoted as  $(e_1, \dots, e_n)$  where  $e_i \in S$  for all  $i$  as long as the initial vertex of the path is specified. In many of our arguments the initial vertex is not important as the argument is valid for such a path at any vertex. As an example, whether or not a path is geodesic does not depend on the initial vertex. In these cases we may suppress initial vertices.

Perhaps the most useful combinatorial and geometric fact about Coxeter groups is the following which is referred to as the *Deletion Condition* in this paper. It is used in this paper in any result that involves the geometry of the Cayley graph  $\Lambda$ .

**Theorem 1.1. Deletion Condition.** *If  $w = w_1 \dots w_n$  is non-geodesic in  $\Lambda$  then there are  $0 < i < j \leq n$  such that  $w = w_1 \dots w_{i-1} w_{i+1} \dots w_{j-1} w_{j+1} \dots w_n$ .*

**Remark.** In this case, we say that  $w_i$  *deletes* with  $w_j$ . Observe that if  $w_1 \dots w_{n-1}$  is geodesic, but  $w_1 \dots w_{n-1} w_n$  is not geodesic, then we know  $w_n$  must delete with one of the previous letters. If the group is right-angled, then  $w_n$  deletes with  $w_i$  for some  $1 \leq i < n$ . In fact,  $w_i = w_n$  and  $i$  is the

largest integer between 1 and  $n - 1$  with  $w_i = w_n$ . Also,  $w_n$  commutes with  $w_j$  for all  $i < j \leq n - 1$ .

**Definition 1.2** (Special subgroup). *Suppose  $(G, S)$  is a Coxeter system and let  $V \subset S$ . Then the subgroup of  $G$  generated by  $V$  is called a special subgroup of  $G$ .*

The next result shows why the special subgroups are indeed special.

**Theorem 1.3. Special Subgroup Theorem.** *If  $(G, S)$  is a Coxeter system and  $V \subset S$  then the subgroup  $\langle V \rangle$  is Coxeter with presentation  $\langle V | R \rangle$  where  $R$  is the set of all relations of  $P(G, S)$  involving only letters from  $V$ .*

The next two lemmas are part of the standard theory of Coxeter groups and proofs can be found in [H] for example. The first of these lemmas says that special subgroups are convex in  $\Lambda(G, S)$ .

**Lemma 1.4.** *If  $U \subset S$  and  $u \in \langle U \rangle$  is written as a geodesic word in  $U$  then this word is geodesic in  $S$ . Furthermore every geodesic for  $u$  can only use letters from  $U$ .*

If  $\alpha$  is a path in  $\Lambda$  then define  $\bar{\alpha} \in G$  to be the product of the edge labels of  $\alpha$ . The following lemma says that if we add one more edge to a geodesic path of length  $n$  in  $\Lambda$ , then the new path cannot also be of length  $n$ . Also, the set of generators that make this new path have length  $n - 1$  only generate a finite special subgroup. This will be very important in lemmas 4.3 and 5.6.

**Lemma 1.5.** *If  $\alpha$  is geodesic in  $\Lambda$  then the set of labels of edges  $e$  such that  $(\alpha, e)$  is not geodesic generates a finite subgroup of  $G$ , denoted  $B(\bar{\alpha})$ . Furthermore, either  $l(\alpha e) = l(\alpha) - 1$  or  $l(\alpha e) = l(\alpha) + 1$*

The next three lemmas are geometric consequences of the convexity of the special subgroups which will be useful in the proof of the main theorem. The first of these follows directly from the Deletion Condition and lemma 1.4. It allows us to construct geodesics in  $\Lambda$  by combining geodesics from special subgroups with non-overlapping generating sets.

**Lemma 1.6.** *Suppose  $(G, S)$  is a Coxeter system,  $U$  and  $V$  are subsets of  $S$  with  $U \cap V = \emptyset$ . If  $\alpha$  is a geodesic in the letters of  $U$  and  $\beta$  is a geodesic in the letters of  $V$ , then  $(\alpha, \beta)$  is a geodesic in  $\Lambda(G, S)$ .*

The next lemma is essentially a more technical way of saying the special subgroups are convex. Specifically, if you start with a word of the form  $sut$  where  $s, t \in S$  with  $t, u$  both in a particular special subgroup (the one generated by the letters of  $u$ ). Then if the edge path  $sut$  ends in the subgroup,  $s$  must also be in that subgroup since  $s$  must equal  $t$  and this generator commutes with the letters of the word  $u$ . This form of the convexity is used in many of the results of this section and in lemmas 4.3 and 5.4.

**Lemma 1.7.** *Suppose  $U \subset S$ . If  $s \in S - U$ ,  $t \in S$  and for some  $u \in \langle U \rangle$ ,  $sut \in \langle U \rangle$ , then*

- (1)  $t = s$ .
- (2)  $sus = u$  and
- (3) If  $u = u_1 \dots u_n$  is geodesic, (so  $u_i \in U$ ), then  $su_i s = u_i$  for all  $i$ .

*Proof.* As  $sut \in \langle U \rangle$ ,  $s \in \langle U \cup \{t\} \rangle$ . By lemma 1.4,  $t = s$  and (1) is finished.

Say  $u = u_1 \dots u_n$  is geodesic, where  $u_i \in U$ . By lemma 1.4, this is geodesic in the entire Coxeter system. Say  $sus = v_1 \dots v_m$ ,  $m$  minimal and  $v_i \in U$ .

If  $su_1 \dots u_n$  were not geodesic, and  $i$  were the first integer such that  $su_1 \dots u_i$  is not geodesic then no two  $u_j$  could be deleted as  $u_1 \dots u_n$  is geodesic. Hence  $s$  and  $u_i$  must delete. But this implies  $s \in \langle U \rangle$ , contradicting lemma 1.4.

Analogously,  $sv_m \dots v_1$  is geodesic and so  $m = n$ .

Now  $su_1 \dots u_n s$  is not geodesic. The second  $s$  cannot delete with a  $u_i$  as  $s \notin \langle U \rangle$ . So  $s$  commutes with  $u$ , finishing (2).

To prove (3), we proceed by induction on  $n$ . If  $n = 1$ , (3) is clear from (2). Assume (3) is true for  $k \leq n - 1$  and  $n \geq 2$ . The path  $u_1 \dots u_n s$  is geodesic (as  $s \notin U$ ), but  $u_1 \dots u_n s u_n (= su_1 \dots u_{n-1})$  is not geodesic. Clearly the last  $u_n$  and  $s$  do not delete as  $s \neq u_n$ . Say the last  $u_n$  and  $u_i$  delete. This implies (by part (2)) that  $s$  commutes with both  $u_1 \dots u_{n-1}$  and  $u_1, \dots, u_{i-1} u_{i+1} \dots u_n$ . By induction,  $s$  commutes with  $u_i$  for all  $i$ .  $\square$

It is an elementary exercise using the deletion condition to show that because of convexity of the special subgroups, we also have the notion of a unique projection onto a special subgroup (or one of its cosets).

**Lemma 1.8.** *Suppose  $*$  and  $v$  are vertices of  $\Lambda$  and  $T \subset S$ . Then there is a unique vertex  $w$  of  $v\langle T \rangle$  closest to  $*$ . Furthermore, if  $\alpha$  is a geodesic from  $*$  to  $w$  and  $\beta$  is a geodesic in the letters of  $T$ , then  $(\alpha, \beta)$  is a geodesic in  $\Lambda$ .*

Lemma 1.8 and the Deletion Condition now imply:

**Lemma 1.9.** *Suppose  $\beta$  is a geodesic from  $x$  to  $y$  in  $\Lambda$  and  $(\beta, s)$  is not geodesic. If  $t$  is the last letter of  $\beta$  then  $s$  is related to  $t$  and the edge path labeled by half of this relation, at  $y$ , is geodesic towards  $x$ .*

We now begin with the combinatorial lemmas which are necessary to prove the Finite Index Lemma below. For those readers interested in the main theorem, skip to the statement of lemma 1.16.

**Definition 1.10** (2-link). *For  $A \subset S$ , we define  $lk^2(A) = \cap_{a \in A} lk^2(a)$  where  $lk^2(a)$  means all  $s \in S$  that are connected to  $a$  in the graph  $\Gamma(G, S)$  with an edge labeled 2. Let  $D(A) = \{g \in G \mid gag^{-1} \in A, \forall a \in A\}$ . In particular, for  $s \in S$ , we often write  $lk^2(s)$  instead of  $lk^2(\{s\})$ .*

We make the following observations concerning the right-angled Coxeter groups. These statements will be used in the last three sections of the paper when we restrict to the right-angled case.

**Remarks.** Let  $(G, S)$  be a right-angled,  $H \subset G$ , let  $C(H)$  denote the centralizer of  $H$  in  $G$ . If  $H = \{s\}$  a singleton set, we write  $C(s)$  instead of  $C(\{s\})$ .

- (1) For  $(G, S)$  right-angled with  $A \subset S$ ,  $lk^2(A) = lk(A)$  where  $lk(A)$  is the usual link of  $A$  in the graph  $\Gamma(G, S)$ .
- (2)  $\langle lk(A) \rangle = D(A)$ .
- (3) For  $s \in S$ ,  $C(s)/\langle s \rangle = \langle lk(s) \rangle$ . This follows from lemma 1.7 and the Deletion Condition. In particular,  $\langle lk(s) \rangle$  is an index two subgroup of  $C(s)$  in this case.

**Lemma 1.11.** *If  $s$  and  $t$  are unrelated generators in a Coxeter presentation, then  $D(s, t) = C(s, t) = \langle lk^2(\{s, t\}) \rangle$ .*

*Proof.* Clearly  $lk^2(\{s, t\}) \subset C(s, t) \subset D(s, t)$ . Suppose that  $w = v_1 \dots v_n$  is in  $D(s, t)$ . If  $s \notin \{v_1, \dots, v_n\}$  and  $ws w^{-1} = a \in S$ , then  $a = s$  and each  $v_i$  is an element of  $lk^2(s)$  by lemma 1.7. Hence if neither  $s$  nor  $t$  is in  $\{v_1 \dots v_n\}$ , each  $v_i \in lk^2(\{s, t\})$  and we are finished.

Without loss of generality, assume the last  $s$  or  $t$  occurring in  $v_1 \dots v_n$  is  $s = v_i$ . Choose  $a \in S$  such that  $wtw^{-1} = a$ .

If  $v_1 \dots v_n t$  is not a geodesic edge path in  $\Lambda$ , then  $t$  is related to  $v_n$  by Lemma 1.8. Also if  $*$  is a base point and  $x$  is the end point of  $(v_1, \dots, v_n)$  and  $(v_n t)^{m_{ij}} = 1$  then the two edge paths labeled by half of this relation at  $x$  are geodesic towards  $*$  by Lemma 1.8. In particular,  $v_1 \dots v_{n-1} t$  is not geodesic and  $t$  is related to  $v_{n-1}$ . Continuing we get  $t$  is related to  $s$ , which is contrary to our hypothesis.

Now if  $v_1 \dots v_n t$  is geodesic, then so is  $av_1 \dots v_n$ . Also  $av_1 \dots v_n t$  is not geodesic. Repeat the above argument on  $av_1 \dots v_n t$  to again get  $t$  and  $s$  related, a contradiction.  $\square$

Suppose  $V \subset S$ . Then  $\langle V \rangle$  visually decomposes (or splits) as a (non-trivial) direct product if  $A \subset V$  such that  $A \neq \emptyset$ ,  $A \neq V$  and  $A \subset lk^2(V - A)$ . In which case  $\langle V \rangle = \langle A \rangle \oplus \langle V - A \rangle$ .

**Proposition 1.12.** *Suppose  $V$  is the vertex set of a complete subgraph of  $\Gamma$ ,  $\langle V \rangle$  is infinite and  $\langle V \rangle$  does not visually decompose as a non-trivial direct product. Then  $D(V) = C(V) = \langle lk^2(V) \rangle$ .*

*Proof.* Clearly  $lk^2(V) \subset C(V) \subset D(V)$ . Suppose  $w \in D(V)$ . Write  $w$  geodesically as  $w_1 \dots w_n$ . Let  $A = V \cap \{w_1, \dots, w_n\}$ . If  $A = \emptyset$ , then by lemma 1.7, each  $w_i$  is in  $lk^2(V)$  and so  $w \in \langle lk^2(V) \rangle$ . (We will in fact show that  $A = \emptyset$ .)

Suppose  $A \neq \emptyset$  and  $A \neq V$ . By lemma 1.7, each  $s \in V - A$  commutes with each  $w_i$ . In particular,  $\langle V - A \rangle$  commutes with  $\langle A \rangle$ , violating our hypothesis.

Hence it suffices to show  $V \neq A$ , i.e.  $V \not\subset \{w_1, \dots, w_n\}$ . Suppose  $V = A$ . Choose  $Y \subset V$  such that  $\langle Y \rangle$  is infinite, but for any  $y \in Y$ ,  $\langle Y - \{y\} \rangle$  is finite. By Lemma 1.5, there is a geodesic ray  $(w_1, \dots, w_n, y_1, y_2, \dots)$  such that  $y_i \in Y$ . From the definition of  $Y$ , for each  $y \in Y$  there are infinitely many  $i$  such that  $y_i = y$ . Choose  $s_i \in S$  such that  $wy_i w^{-1} = s_i$ .

**Claim 1.**  $w_n \notin Y$ .

*Proof of Claim 1.* The edge path  $(s_1, \dots, s_{i-1}, w_1, \dots, w_n, y_i, \dots)$  is geodesic for all  $i$ . Hence  $y_i \neq w_n$  for all  $i$ . This completes Claim 1.

**Claim 2.** The generator  $w_n (\notin Y)$  commutes with each  $y \in Y$  (so  $w_1 \dots w_{n-1} \in D(Y)$ ) and  $(w_1, \dots, w_{n-1}, y_1, y_2, \dots)$  is geodesic.

*Proof of Claim 2.* As  $wy_iw^{-1} = a_i$  for all  $i$ ,  $(w_1, \dots, w_n, y_1, \dots, y_i, w_n)$  is not geodesic.

Since  $w_n \notin Y$ , the last  $w_n$  does not delete with  $y_i$ . Hence  $w_n$  deletes with some edge of  $(w_1, \dots, w_n)$ . Select integers  $0 < i_1 < i_2 < \dots < i_{n+1}$  such that each letter of  $Y$  is in  $\{y_{i_j+1}, \dots, y_{i_{j+1}}\}$ , for each  $j$ . There must be  $j < h$  such that the  $w_n$ -edge ending  $(w_1, \dots, w_n, y_1, \dots, y_{i_j}, w_n)$  and  $(w_1, \dots, w_n, y_1, \dots, y_{i_h}, w_n)$  delete with the same letter, say  $w_k$  of  $w$ . Then  $w_k w_{k+1} \dots w_n y_1 \dots y_{i_j} = w_{k+1} \dots w_n y_1 \dots y_{i_j} w_n$  and  $w_k w_{k+1} \dots w_n y_1 \dots y_{i_j} \dots y_{i_h} w_n = w_{k+1} \dots w_n y_1 \dots y_{i_j} \dots y_{i_h}$ . Hence,  $w_{k+1} \dots w_n y_1 \dots y_{i_j} w_n y_{i_j+1} \dots y_{i_h} w_n = w_{k+1} \dots w_n y_1 \dots y_{i_j} \dots y_{i_h}$  and  $w_n y_{i_j+1} \dots y_{i_h} w_n = y_{i_j+1} \dots y_{i_h}$ . By lemma 1.7,  $w_n$  commutes with each  $y \in Y$ , finishing Claim 2.

Using Claim 1, we now see that  $w_{n-1} \notin Y$ . Continuing we have  $\{w_1, \dots, w_n\} \cap Y = \emptyset$ . In particular  $V \not\subset \{w_1, \dots, w_n\}$ .  $\square$

**Lemma 1.13.** *Suppose  $V$  is a non-empty subset of  $S$  and  $D(V) = \langle lk^2(V) \rangle$  (respectively  $C(V) = \langle lk^2(V) \rangle$ ). If  $s \in S - V$  and  $s \notin lk^2(V)$  then  $D(V \cup \{s\}) = \langle lk^2(V \cup \{s\}) \rangle$ . (Respectively,  $C(V \cup \{s\}) = \langle lk^2(V \cup \{s\}) \rangle$ ).*

*Proof.* The proof is essentially the same in both cases. Suppose  $w \in D(V \cup \{s\}) - \{1\}$  is written geodesically as  $w_1 \dots w_n$ . As  $w \in D(V)$ ,  $w_i \in lk^2(V)$ . Hence no  $w_i = s$ . By lemma 1.7, each  $w_i$  is an element of  $lk^2(s)$  and so each  $w_i$  is an element of  $lk^2(V \cup \{s\})$ .  $\square$

**Lemma 1.14.** *Suppose  $(G, S)$  is a Coxeter system,  $V \subset S$ ,  $\langle V \rangle$  is infinite and  $F$  is the (unique) maximal subset of  $S$  such that  $\langle F \rangle$  is finite and  $\langle V \rangle$  decomposes as  $\langle F \rangle \oplus \langle V - F \rangle$ . Then  $D(V) = \langle lk^2(V - F) \rangle \cap D(F)$  and  $C(V) = \langle lk^2(V - F) \rangle \cap C(F)$ .*

*Proof.* Again the proof is essentially the same in each case. As  $D(V) = D(F) \cap D(V - F)$  it suffices to show  $D(V - F) = \langle lk^2(V - F) \rangle$ . Choose  $V_1, \dots, V_n$  subsets of  $V - F$  such that  $V - F = \cup_{i=1}^n V_i$ ,  $\langle V - F \rangle = \oplus_{i=1}^n \langle V_i \rangle$  and  $n$  is maximal. Note that  $\langle V_i \rangle$  is infinite for all  $i$ . If  $V_i$  is complete, then by Proposition 1.10,  $D(V_i) = lk^2(V_i)$ . If  $V_i$  is not complete, say  $s, t \in V_i$  are unrelated in  $P$ . By Lemma 1.9,  $D(s, t) = lk^2(s, t)$  and Theorem 1.11 implies that  $D(V_i) = lk^2(V_i)$ . Now  $D(V - F) = \cap_{i=1}^n D(V_i) = \cap_{i=1}^n \langle lk^2(V_i) \rangle = \langle lk^2(V - F) \rangle$ .  $\square$

**Lemma 1.15.** *If  $V$  does not visually split non-trivially as a direct product, then for any geodesic  $\alpha$  in  $V$  and  $v \in V$ ,  $\alpha$  can be extended to a geodesic ending with  $v$ .*

*Proof:* If not, choose  $\alpha$  a geodesic such that  $Y_\alpha = \{y \in S \mid \alpha \text{ cannot be extended to a geodesic ending with } y\}$  is as large as possible. By hypothesis,  $Y_\alpha \neq \emptyset$  and by lemma 1.5,  $Y_\alpha \neq S$ . By the maximality of  $Y_\alpha$ ,  $\alpha$  can be extended to an infinite geodesic  $(\alpha, \gamma)$  such that the letters of  $S - Y_\alpha$  occur infinitely often in  $\gamma$ . We show  $Y_\alpha$  commutes with  $S - Y_\alpha$  to obtain a contradiction.

Assume  $\alpha = (a_1, \dots, a_n)$  and  $\gamma = (b_1, b_2, \dots)$ . Let  $\beta_i = (\alpha, b_1, b_2, \dots, b_i)$ . For  $y \in Y_\alpha$ ,  $(\beta_i, y)$  is not geodesic. This last  $y$  cannot delete with a  $b_j$  as  $y \notin S - Y_\alpha$  and so must delete with some  $a_k$ . Choose  $i < j$  such that all letters of  $S - Y_\alpha$  occur in  $\{b_{i+1}, \dots, b_j\}$  and the  $y$  ending  $(\beta_i, y)$  and  $(\beta_j, y)$  both delete with  $a_k$ . Then  $a_k \dots a_n b_1 \dots b_i = a_{k+1} \dots a_n b_1 \dots b_i y$  and  $a_k \dots a_n b_1 \dots b_j y = a_{k+1} \dots a_n b_1 \dots b_j$ . So  $a_{k+1} \dots a_n b_1 \dots b_i y b_{i+1} \dots b_j y = a_{k+1} \dots a_n b_1 \dots b_j$  and  $y b_{i+1} \dots b_j y = b_{i+1} \dots b_j$ . By lemma 1.7,  $y$  commutes with  $S - Y_\alpha$  as needed.  $\square$

**Lemma 1.16. Finite Index Lemma** *Suppose  $(G, S)$  is a Coxeter system,  $G$  is infinite,  $U \subset S$ ,  $\langle U \rangle$  and has finite index in  $G$  and  $F \subset U$  is the maximal set such that  $\langle F \rangle$  is finite and  $U$  visually splits as  $\langle F \rangle \oplus \langle U - F \rangle$  then  $\langle (S - U) \cup F \rangle$  is finite and  $G$  splits as  $\langle (S - U) \cup F \rangle \oplus \langle U - F \rangle$ .*

*Proof:* Clearly  $\langle U - F \rangle$  has finite index in  $V$ . Hence there is a bound  $K$  such that every vertex of  $\Lambda(G, S)$  is within  $K$  of  $\langle U - F \rangle$ . If  $\langle (S - U) \cup F \rangle$  were not finite, choose a geodesic  $\alpha$  of length  $K + 1$  in the letters  $(S - U) \cup F$ , with initial point the identity. The end point  $y$  of this geodesic is within  $K$  of a vertex  $x \in \langle U - F \rangle$ . Let  $\beta$  be a geodesic in the letters of  $U - F$  from  $x$  to the identity. By the lemma 1.6,  $(\beta, \alpha)$  is a geodesic from  $x$  to  $y$ , but  $d(x, y) \leq K$ , giving the desired contradiction. So  $\langle (S - U) \cup F \rangle$  is finite.

Maximally decompose  $\langle U - F \rangle$  as  $\bigoplus_{i=1}^n \langle W_i \rangle$  with  $W_i \neq \emptyset$ ,  $\cup_{i=1}^n W_i = U - F$ , and  $W_i \cap W_j = \emptyset$  for  $i \neq j$ . By the lemma 1.15, there is a geodesic  $\alpha_i$  with letters in  $W_i$  that begins at  $1 \in \Lambda$  such that each letter of  $W_i$  occurs infinitely often in  $\alpha_i$ . Let  $\beta$  be an edge path at  $1$  such that each element of  $(S - U) \cup F$  labels exactly one edge of  $\beta$  and  $\beta$  has length  $\text{Card}((S - U) \cup F)$ . By lemma 1.4,  $\beta$  is geodesic. Say  $\alpha_i = (a_{i1}, a_{i2}, \dots)$  and  $\beta = (b_1, \dots, b_n)$ . Choose  $k$  such that all letters of  $W_i$  occur at least  $n + K + 1$  times in  $\alpha_{ik} = (a_{i1}, \dots, a_{ik})$ . Let  $x$  be the end point of  $\beta$  and  $y$  the end point of  $\alpha_{ik}$ . Let  $\delta$  be a geodesic from  $y$  to  $x \langle U - F \rangle$  of length  $\leq K$ . Say  $\delta$  ends at  $z$ . Let  $\gamma = (c_1, \dots, c_m)$  be a geodesic from  $x$  to  $z$  with  $c_j \in (U - F)$  for all  $j$ . By the lemma 1.6  $(\alpha_i^{-1}, \beta)$  is geodesic and by lemma 1.8,  $(\delta, \gamma^{-1})$  is also geodesic. Hence  $(\alpha_{ik}^{-1}, \beta, c_1)$  is not geodesic.

Say  $c_1$  deletes with  $a_{ij}$ . Since  $b_i \notin U - F$  for all  $i$ , we must have  $c_1$  commuting with  $b_j$  for all  $j$  by lemma 1.7. From the Deletion Condition we see that if  $(u_1, \dots, u_n)$  and  $(v_1, \dots, v_n)$  are geodesics with the same endpoints, then  $\{u_1, \dots, u_n\} = \{v_1, \dots, v_n\}$ , thus we obtain  $\{c_1, \dots, c_m\} \subset U - F$ . If we replace  $\alpha_i$  with a geodesic in  $U - F$  from  $c_1$  to  $y$ , we see that  $c_2$  commutes with each  $b_j$ . Continuing, each  $b_j$  commutes with each  $c_q$ . Now  $c_1 \dots c_m = a_{i1} \dots a_{ik} \delta \beta^{-1}$ . At most  $n + K$  deletions can occur in the right

hand side before a geodesic is obtained. Hence all letters of  $U - F$  appear in the resulting reduced word. So all letters of  $U - F$  appear in  $\{c_1 \dots, c_m\}$  forcing the letters of  $U - F$  to commute with the letters of  $(S - U) \cup F$  as needed.  $\square$

## 2. CAT(0) SPACES AND THEIR BOUNDARIES

In this section we give definitions and basic properties of CAT(0) spaces, boundaries and isometries as well as some known facts we will need in the proof of the main result.

Let  $(X, d)$  be a metric space. Then  $X$  is *proper* if closed metric balls are compact. A (*unit speed*) *geodesic* from  $x$  to  $y$  for  $x, y \in X$  is a map  $c : [0, D] \rightarrow X$  such that  $c(0) = x$ ,  $c(D) = y$  and  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, D]$ . If  $I \subseteq \mathbb{R}$  then a map  $c : I \rightarrow X$  parametrizes its image *proportional to arclength* if there exists a constant  $\lambda$  such that  $d(c(t), c(t')) = \lambda|t - t'|$  for all  $t, t' \in I$ . Lastly,  $(X, d)$  is called a *geodesic metric space* if every pair of points are joined by a geodesic.

**Definition 2.1** (CAT(0)). *Let  $(X, d)$  be a proper complete geodesic metric space. If  $\Delta abc$  is a geodesic triangle in  $X$ , then we consider  $\Delta \bar{a}\bar{b}\bar{c}$  in  $\mathbb{E}^2$ , a triangle with the same side lengths, and call this a comparison triangle. Then we say  $X$  satisfies the CAT(0) inequality if given  $\Delta abc$  in  $X$ , then for any comparison triangle and any two points  $p, q$  on  $\Delta abc$ , the corresponding points  $\bar{p}, \bar{q}$  on the comparison triangle satisfy*

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

If  $(X, d)$  is a CAT(0) space, then the following basic properties hold:

- (1) The distance function  $d: X \times X \rightarrow \mathbb{R}$  is convex.
- (2)  $X$  has unique geodesic segments between points.
- (3)  $X$  is contractible.

For details, see [BH].

Let  $(X, d)$  be a proper CAT(0) space. First, define the boundary,  $\partial X$  as a set as follows:

**Definition 2.2** (Asymptotic). *Two geodesic rays  $c, c' : [0, \infty) \rightarrow X$  are said to be asymptotic if there exists a constant  $K$  such that  $d(c(t), c'(t)) \leq K, \forall t > 0$  - this is an equivalence relation. The boundary of  $X$ , denoted  $\partial X$ , is then the set of equivalence classes of geodesic rays. The union  $X \cup \partial X$  will be denoted  $\bar{X}$ . The equivalence class of a ray  $c$  is denoted by  $c(\infty)$ .*

There is a natural neighborhood basis for a point in  $\partial X$ . Let  $c$  be a geodesic ray emanating from  $x_0$  and  $r > 0$ ,  $\epsilon > 0$ . Also, let  $S(x_0, r)$  denote the sphere of radius  $r$  centered at  $x_0$  with  $p_r : X \rightarrow S(x_0, r)$  denoting projection. Define

$$U(c, r, \epsilon) = \{x \in \bar{X} \mid d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon\}$$

This consists of all points in  $\bar{X}$  such that when projected back to  $S(x_0, r)$ , this projection is not more than  $\epsilon$  away from the intersection of that sphere

with  $c$ . These sets along with the metric balls about  $x_0$  form a basis for the cone topology. The set  $\partial X$  with the cone topology is often called the *visual boundary*. As one expects, the visual boundary of  $\mathbb{R}^n$  is  $S^{n-1}$  as is the visual boundary of  $\mathbb{H}^n$ .

**Definition 2.3.** *Let  $\gamma$  be an isometry of the metric space  $X$ . The displacement function  $d_\gamma: X \rightarrow R_+$  defined by  $d_\gamma(x) = d(\gamma \cdot x, x)$ . The translation length of  $\gamma$  is the number  $|\gamma| = \inf\{d_\gamma(x) : x \in X\}$ . The set of points where  $\gamma$  attains this infimum will be denoted  $\text{Min}(\gamma)$ . An isometry  $\gamma$  is called semi-simple if  $\text{Min}(\gamma)$  is non-empty.*

We summarize some basic properties about this  $\text{Min}(\gamma)$  in the following proposition.

**Proposition 2.4.** *Let  $X$  be a metric space and  $\gamma$  an isometry of  $X$ .*

- (1)  *$\text{Min}(\gamma)$  is  $\gamma$ -invariant.*
- (2) *If  $\alpha$  is another isometry of  $X$ , then  $|\gamma| = |\alpha\gamma\alpha^{-1}|$ , and  $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$ ; in particular, if  $\alpha$  commutes with  $\gamma$ , then it leaves  $\text{Min}(\gamma)$  invariant.*
- (3) *If  $X$  is CAT(0), then the displacement function  $d_\gamma$  is convex: hence  $\text{Min}(\gamma)$  is a closed convex subset of  $X$ .*

**Definition 2.5.** *Let  $X$  be a metric space. An isometry  $\gamma$  of  $X$  is called*

- (1) *elliptic if  $\gamma$  has a fixed point - i.e  $|\gamma| = 0$  and  $\text{Min}(\gamma)$  is non-empty.*
- (2) *hyperbolic if  $d_\gamma$  attains a strictly positive infimum.*
- (3) *parabolic if  $d_\gamma$  does not attain its infimum, in other words if  $\text{Min}(\gamma)$  is empty.*

It is clear that an isometry is semi-simple if and only if it is elliptic or hyperbolic. If two isometries are conjugate in  $\text{Isom}(X)$ , then they are in the same class.

If a group  $\Gamma$  acts geometrically on a CAT(0) space  $X$ , then the elements of  $\Gamma$  act as semi-simple isometries because of the cocompactness of the action.

The next theorem implies that the centralizer of an element  $\gamma$  in a CAT(0) group  $\Gamma$  is again a CAT(0) group because it acts geometrically on CAT(0) subspace  $\text{Min}(\gamma)$ . A proof of this can be found in [R].

**Theorem 2.6.** *Suppose  $\Gamma$  acts geometrically on the CAT(0) space  $X$  and suppose  $\gamma \in \Gamma$ . Then  $C_\Gamma(\gamma)$  acts geometrically on the CAT(0) subset  $\text{Min}(\gamma)$  of  $X$ .*

Remarks. We make the following remarks which will be needed in our setting of right-angled Coxeter groups.

- (1) If  $(G, S)$  is a right-angled Coxeter group acting geometrically on the CAT(0) space  $X$ , then each  $s \in S$  is an elliptic isometry with  $\text{Min}(s) = \text{Fix}(s)$  a closed, convex subset of  $X$  on which the centralizer of  $s$  acts geometrically.

- (2) Since  $C(s)$  acts geometrically on  $\text{Fix}(s)$  and  $\langle lk(s) \rangle$  is an index two subgroup of  $C(s)$  (see remarks following definition 1.10) thus  $\langle lk(s) \rangle$  is a  $CAT(0)$  subgroup of  $G$ .

We will need the following result concerning  $CAT(0)$  products. Proofs of these can be found in [BH].

**Theorem 2.7.** *Suppose  $X$  and  $Y$  are  $CAT(0)$  spaces. Then  $X \times Y$  with the product metric is also a  $CAT(0)$  space. Furthermore,  $\partial(X \times Y)$  is homeomorphic to the spherical join of  $\partial X$  and  $\partial Y$ . In particular, if  $Y = \mathbb{R}$ , then  $\partial(X \times \mathbb{R})$  is homeomorphic to the suspension of  $\partial X$ .*

The following theorem is due to Milnor in [M] and will be used throughout the paper to carry information between the Cayley graph of  $(G, S)$  and the  $CAT(0)$  space  $X$ .

**Theorem 2.8.** *If a group  $G$  with finite generating set  $S$  acts geometrically on a proper, geodesic metric space  $X$ , then the Cayley graph of  $G$  with respect to  $S$  is quasi-isometric to  $X$  under the map  $g \mapsto g \cdot x_0$  where  $x_0$  is a fixed basepoint in  $X$ .*

### 3. LOCAL CONNECTIVITY

This section contains the proof of Theorem 3.2 below. The material in this section is not needed in the proof of the main theorem. We say a  $CAT(0)$  group  $G$  has *(non-)locally connected boundary* if for every  $CAT(0)$  space  $X$  on which  $G$  acts geometrically,  $\partial X$  is (non-)locally connected. Clearly, if  $G$  is infinite ended, then  $G$  has non-locally connected boundary, thus we only need to consider Coxeter groups  $(G, S)$  for which the presentation graph  $\Gamma$  is connected.

**Definition 3.1 (VFS).** *Suppose  $(G, S)$  is a Coxeter system. Let  $\Gamma$  denote the presentation graph for  $(G, S)$ . A virtual factor separator (VFS) for  $\Gamma$  is a triple  $(C, C_1, K)$  where  $C_1 \subset C$  are full subgraphs of  $\Gamma$ ,  $C$  separates  $\Gamma$ , the group  $\langle C_1 \rangle$  has finite index in  $\langle C \rangle$ ,  $K \subset lk^2(C_1)$ , and  $\langle K \rangle$  is infinite. If  $S = C \cup \{s, t\}$ ,  $s$  and  $t$  do not span an edge of  $\Gamma$  and  $s, t \in lk^2(C)$ , then  $C$  is called a suspended separator of  $\Gamma$ . Note that if  $C$  is a suspended separator of  $\Gamma$ , then  $G = \langle C \rangle \oplus \langle s, t \rangle$  and  $(C, C, \{s, t\})$  is a VFS.*

**Theorem 3.2.** *Suppose  $(G, S)$  is a Coxeter group with presentation graph  $\Gamma$ .*

- (1) *If  $\Gamma$  has a suspended separator  $C$ , then  $G$  has locally connected boundary if and only if  $\langle C \rangle$  has locally connected boundary.*
- (2) *If  $\Gamma$  has a VFS  $(C, C_1, K)$  and  $C$  is not a suspended separator, then  $G$  has non-locally connected boundary.*

The proof of Theorem 3.2 will be an application of the following Theorem from [MR]. The theorem cannot be applied directly because of condition three below. The necessary changes to the original proof are given here.

**Theorem 3.3.** *Suppose  $A$ ,  $B$  and  $C$  are finitely generated groups and  $G = A *_C B$  acts geometrically on a  $CAT(0)$  space  $X$ . If the following conditions are satisfied, then  $\partial X$  is not locally connected:*

- (1)  $[A : C] \geq 2$ ,  $[B : C] \geq 3$ .
- (2) There exists  $s \in G - C$  with  $s^n \notin C$  for all  $n \neq 0$  and  $sCs^{-1} \subset C$ .
- (3)  $Cx_0$  is quasi-convex in  $X$  for a basepoint  $x_0$ .

**Remark.** If  $C_1$  is a subgroup of finite index in  $C$ , and condition (2) is replaced by (2') below then the proof remains unchanged.

(2') There exists  $s \in G - C_1$  with  $s^n \notin C$  for all  $n \neq 0$  and  $sC_1s^{-1} \subset C_1$ .

*Proof of Theorem 3.2*

If  $C$  is a suspended separator, then  $G$  splits as  $\langle C \rangle \oplus \langle \{s, t\} \rangle$ . In this case, the element  $(st)$  is virtually central -meaning, it is central in a subgroup of finite index in  $G$ . In fact, the subgroup generated by  $C$  and the element  $(st)$  is a subgroup of finite index in  $G$  in which  $st$  is central. Theorem 3.4 of [R] implies that if  $X$  is any  $CAT(0)$  space on which  $G$  acts geometrically, then the following are true. A subset  $Z$  of  $X$  is *quasi-dense* if there exists a constant  $K \geq 0$  such that every  $x \in X$  is within  $K$  of an element of  $Z$ .

- (1) There exists a quasi-dense closed, convex subset  $Z$  of  $X$  such that  $\partial Z = \partial X$  (we know  $\partial Z \subset \partial X$  since  $Z$  convex in  $X$ . Thus for boundary considerations, we only need to know about the space  $Z$ ).
- (2)  $Z$  splits isometrically as  $Y \times \mathbb{R}$  where  $Y$  is a closed, convex subset of  $X$  and  $Y$  admits a geometric group action by the group  $\langle C \rangle$ .

Since  $Z$  splits as a product, we know  $\partial Z = \partial X \equiv \Sigma(\partial Y)$  by Theorem 2.7. Thus  $\partial X$  is locally connected if and only if  $\partial Y$  is locally connected.

Notice that if  $Y$  is a  $CAT(0)$  space on which  $\langle C \rangle$  acts geometrically, then we can construct the  $CAT(0)$  space  $X = Y \times \mathbb{R}$  and a geometric action of  $G$  on  $X$  via the product action. Simply let  $\langle C \rangle$  act on  $Y$  as given and let  $\langle \{s, t\} \rangle$  act by the infinite dihedral action on the  $\mathbb{R}$  factor. This finishes part 1 of the theorem.

Suppose  $(C, C_1, K)$  is a VFS for  $\Gamma$ . Then  $G = \langle A \rangle *_C \langle B \rangle$  where  $A \cup B = S$  and  $A \cap B = C$  since  $C$  separates  $\Gamma$ . The proof of this theorem is not a direct application of Theorem 3.3 since it is not clear that  $\langle C \rangle$  is quasi-convex in an arbitrary  $CAT(0)$  space on which  $G$  acts.

There are two cases to consider. Since  $\langle K \rangle$  is infinite, either  $K$  contains two unrelated elements  $s_1, s_2$  of  $S$  or  $K$  is complete.

Suppose  $s_1, s_2 \in K$  are unrelated. If  $s_1$  and  $s_2$  are in different components of  $\Gamma - C$ , then the only vertices of  $S - C_1$  that commute with both  $s_1$  and  $s_2$  are in  $C - C_1$ . In this case, we see that  $C_1 \subset lk^2(s_1, s_2) \subset C$  and  $S \neq lk^2(s_1, s_2) \cup \{s_1, s_2\}$  since  $C$  is not a suspended separator of  $\Gamma$ . Now  $\langle lk^2(s_1, s_2) \rangle$  has finite index in  $\langle C \rangle$ . By Theorem 2.6  $\langle lk^2(s_1, s_2) \rangle$  (and hence  $\langle C \rangle$ ) is quasi-convex in any  $CAT(0)$  space on which  $G$  acts. Thus we can directly apply Theorem 3.3 (with (2) replaced by (2')) to this case.

Note that not both  $s_1$  and  $s_2$  are in  $C$ . Otherwise a power of the element  $(s_1s_2)$  would be in  $\langle C_1 \rangle$  which would imply both  $s_1$  and  $s_2$  are in  $C_1$  by lemma 1.4. If  $s_1$  and  $s_2$  are in the same component of  $\Gamma - C$  or say  $s_2$  is in  $C - C_1$  and  $s_1 \in \Gamma - C$ , then we may assume  $b$  is a vertex in a component of  $\Gamma - C$  other than the component containing  $s_1$ . Since  $s_1 \in K$ ,  $b$  is not in  $lk^2(K)$ . Also, if  $K$  is complete, then not all of the vertices of  $K$  are in  $C - C_1$ . This is because  $\langle K \rangle$  is infinite and (by Lemma 1.13)  $\langle C - C_1 \rangle$  is finite. But all of the vertices of  $K - C$  are in the same component of  $\Gamma - C$ . In order to consider the cases  $K$  complete and  $K$  not complete at the same time, we let  $s_1$  be an arbitrary element of  $K - C$  and  $b$  an element of a component of  $\Gamma - C$  other than the component containing  $s_1$  (so again,  $b$  is not in  $lk^2(K)$ ).

The proof of our theorem follows that of [MR] but several adjustments (which we point out below) are necessary. There are three places where the quasiconvexity of  $C$  in  $X$  is used in the original proof: Lemma 3.6, Lemma 3.7 and early in §4 and these produce the most extensive changes.

Let  $s$  be an element of infinite order in  $\langle K \rangle$  (we can do this because  $\langle K \rangle$  is infinite). Let  $x_0$  be a fixed basepoint in  $X$ . Our goal (following [MR]) is to show that  $s(\infty)$  is not in the limit set of  $\langle lk^2(K) \rangle \cdot x_0$  and to construct a sequence of geodesic rays  $u_i$  in  $X$  so that  $u_i \rightarrow s(\infty)$  and any path in  $\partial X$  from  $u_i$  to  $u_j$  passes through the limit set of  $\langle lk^2(K) \rangle$ .

As  $\langle C_1 \rangle$  has finite index in  $\langle C \rangle$ , the limit set of  $\langle C_1 \rangle \cdot x_0$  (denoted  $L(C_1)$ ) is the same as  $L(C)$ . So  $L(C) = L(C_1) \subset L(lk^2(K))$ . Now observe that  $G$  visually splits as  $\langle A \rangle *_{\langle C \rangle} \langle B \rangle$  where  $K \subset A$ ,  $b \in B$ . Note that for  $k \neq 0$ ,  $s^k \notin \langle C_1 \rangle$  (since  $C_1 \subset lk^2(K)$ ) and so  $s^k \notin C$ . Thus  $\langle C \rangle$  has infinite index in  $\langle A \rangle$  and index at least 2 in  $\langle B \rangle$ .

Lemmas 3.1 and 3.2 of [MR] are unchanged with  $C$  replaced by  $\langle C \rangle$ .

The ray  $r_0$  defined following Lemma 3.2 will be  $\langle b, s_1, b, s_1, \dots \rangle$  in our setting. With this choice, Lemmas 3.3 and 3.4 go unchanged.

Replace  $Q$ , the quasi-convexity constant for  $Cx_0$ , by the quasiconvexity constant for  $\langle lk^2(K) \rangle x_0$ . As stated above,  $s^k \notin \langle lk^2(K) \rangle$  for all  $k \neq 0$ .

Lemma 3.6 can be replaced with:

**Lemma\*3.6.** The set  $L(lk^2(K)) \subset \partial X$  does not contain  $s(\infty)$ .

In Lemma 3.7 replace  $C$  by  $lk^2(K)$ , and the proof is unchanged.

We are now ready to construct the  $u_i$  of §4 [MR]. Select  $t_1 < t_2$  exactly as in [MR]. We must be more careful with the selection of  $\bar{t}$ . First observe that if  $u$  and  $v$  are different vertices of  $r_0 \equiv \langle b, s_1, b, s_1, \dots \rangle$  then  $u \langle lk^2(K) \rangle \cap v \langle lk^2(K) \rangle = \emptyset$ . Next we show the following:

**Claim.** The intersection  $u \langle lk^2(K) \rangle \cap v \langle C \rangle = \emptyset$ .

If not, suppose  $x \in u \langle lk^2(K) \rangle \cap v \langle C \rangle$ . By Lemma 1.15, the sub-geodesic of  $r_0$  from  $u$  to  $v$  followed by a geodesic in  $v \langle lk^2(K) \rangle$  from  $v$  to  $x$  is geodesic. Hence any geodesic from  $u$  to  $x$  in the letters of  $C$  would use the same letters as the previous geodesic by lemma 1.4. But neither  $b$  nor  $s_1$  is in  $C$ .  $\square$ .

**Lemma\*3.1.** If  $t, u, v$  are vertices of  $r_0$  and  $u$  is between  $t$  and  $v$  then any edge path from  $t \langle lk^2(K) \rangle$  to  $v \langle lk^2(K) \rangle$  must pass through  $u \langle C \rangle$ .

*Proof of Lemma \*3.1.* By Lemma 3.1 of [MR], any path from  $t$  to  $v$  must pass through  $u\langle C \rangle$ . If there were a path from  $x \in t\langle lk^2(K) \rangle$  to  $y \in v\langle lk^2(K) \rangle$  that did not intersect  $u\langle C \rangle$ , the one could take a path in  $t\langle lk^2(K) \rangle$  from  $x$  to  $t$  and a path in  $v\langle lk^2(K) \rangle$  from  $y$  to  $v$ . Combining these three paths gives a path from  $t$  to  $v$  that does not intersect  $u\langle C \rangle$ , the desired contradiction.  $\square$ .

This gives an analogue to Lemma 3.2 with  $v$  and  $w$  of Lemma 3.2 replaced by vertices of  $r_0$  and  $C$  replaced by  $lk^2(K)$ . As  $C_1 \subset lk^2(K)$  and  $\langle C_1 \rangle$  has finite index in  $\langle C \rangle$ , there is an integer  $W$  such that  $\langle C \rangle x_0 \subset N_W(\langle lk^2(K) \rangle x_0)$ .

The above analogue can be used to select  $\bar{t} > t_2$  so that

$$y = qr_0(\bar{t}) \notin N_{E+M+Q+W}(r_0(t_1)\langle lk^2(K) \rangle x_0)$$

(as in [MR]) to obtain

$$d_X(s^n y, r_n(t_1)\langle lk^2(K) \rangle x_0) > E + M + Q + W \text{ and}$$

$$d_X(u_n(\bar{t}), r_n(t_1)\langle lk^2(K) \rangle x_0) > M + Q + W.$$

Our new version of Lemma 3.7 implies

$$u_n([\bar{t}_n, \infty)) \cap N_{M+W}(r_n(t_1)\langle lk^2(K) \rangle x_0) = \emptyset$$

(in particular there is no intersection with  $N_M(r_n(t_1)\langle C \rangle x_0)$  either). The remainder of the proof is exactly as in [MR] except for a change in the final sentence:

For  $m < n$ , any path in  $\partial X$  between  $u_n$  and  $u_m$  must intersect  $L(N_M(\langle C \rangle x_{n-1} \cup \langle C \rangle x_n)) \subset L(lk^2(K))$  as needed.  $\square$

#### 4. TWO IMPORTANT LEMMAS

For the remainder of the paper, we will only be dealing with right-angled Coxeter groups.

In this section, we prove two important lemmas about a right-angled Coxeter group  $G$  acting on a  $CAT(0)$  space  $X$  that will allow us to transfer combinatorial information about the Cayley graph of  $G$  to the space  $X$ . The first is lemma 4.2 which allows us to approximate  $CAT(0)$  geodesic rays with Cayley graph rays. The second is lemma 4.4 which says that if two  $CAT(0)$  geodesic rays are sufficiently close, then the Cayley graph approximating rays can be chosen to have a long common subpeice. The method of proof used here does not extend to give the corresponding results for general Coxeter groups.

Let  $\Gamma(G, S)$  denote the presentation graph for  $G$  with respect to the generating set  $S$  and  $\Lambda(G, S) = \Lambda$  denote the Cayley graph of  $G$  with respect to the generating set  $S$ . We know each  $s \in S$  is order 2 and has a closed, convex, fixed point set in  $X$  denoted by  $Fix(s)$ . With a slight abuse of notation, we also denote the fixed point set of  $s$  in  $\Lambda$  by  $Fix(s)$ . Recall this consists of the midpoints of edges labeled by  $s$  with each endpoint in  $\langle lk(s) \rangle$ .

Let  $x_0$  be a basepoint in  $X$ . We identify a quasi-isometric copy of the Cayley graph  $\Lambda$  of  $G$  inside  $X$  with the orbit of  $x_0$  under the action of  $G$

using Theorem 2.8. We will use  $CAT(0)$  geodesics between adjacent vertices in the Cayley graph. With this identification, any geodesic  $\alpha$  in  $\Lambda$  is assigned a peicewise  $CAT(0)$  geodesic in  $X$  which we denote by  $\alpha_X$ .

We stop to make an important observation about geodesics in  $\Lambda$ .

**Remark.** Suppose  $\alpha = (e_1, e_2, \dots, e_n)$  with each  $e_i \in S$  is an edge path in  $\Lambda$ . The path  $\alpha$  crosses a set of fixed point sets or *walls* as the letters of  $\alpha$  are traversed one by one. These walls are (in order):

$Fix(e_1), Fix(e_1e_2e_1), \dots, Fix(e_1e_2 \cdots e_{n-1}e_n e_{n-1} \cdots e_2e_1)$ .

In other words, at the  $i$ th step,  $\alpha$  crosses the wall  $e_1 \cdots e_{n-1}Fix(e_i)$ . We refer to this wall as an  $e_i$ -wall. Notice that  $\alpha$  can have the same letter  $e_i$  occuring twice which gives (possibly) two different  $e_i$ -walls that  $\alpha$  crosses. With this in mind, the Deletion Condition can then be interpreted geometrically as: the edge path  $\alpha$  is geodesic in  $\Lambda$  if and only if it crosses each wall at most once.

**Definition 4.1** (Tracking). *Suppose  $r : [a, b] \rightarrow X$  is a geodesic segment in  $X$  with  $r(a) = x$  and  $r(b) = y$  and suppose  $\delta > 0$ . We say the Cayley graph geodesic  $\alpha$ ,  $\delta$ -tracks  $r$  (or more precisely, the image of  $r$ ) if every point of  $\alpha_X$  is within  $\delta$  of a point of the image of  $r$  and the endpoints of  $r$  and  $\alpha_X$  are within  $\delta$  of each other.*

**Lemma 4.2.** *There exists a  $\delta > 0$  such that for any geodesic ray  $r : [0, \infty) \rightarrow X$  based at  $x_0$ , there exists a geodesic ray  $\alpha_r$  in  $\Lambda(G, S)$  that  $\delta$ -tracks  $r$ .*

*Proof.* First assume  $r : [0, d] \rightarrow X$  is a unit speed parametrization of the geodesic segment in  $X$  between  $x_0$  and  $g \cdot x_0$  for some  $g \in G$ . Since  $G$  acts cocompactly on  $X$ , there exists a  $K > 0$  such that any point  $x \in X$  is within  $K$  of an orbit point. For each  $i = 1, 2, \dots, D = \lfloor d \rfloor$  (i.e.  $D$  is the largest integer less than or equal to  $d$ ) choose an element  $g_i \in G$  such that  $d(r(i), g_i \cdot x_0) < K$ . Choose Cayley graph geodesics  $\alpha_0$  from  $1_G$  to  $g_1$ ,  $\alpha_1$  from  $g_1$  to  $g_2, \dots, \alpha_D$  from  $g_D$  to  $g$ . Call the peicewise geodesic  $\alpha$ . In  $X$ , the concatenated the paths corresponding to the  $\alpha_i$  to form a peicewise Cayley graph geodesic  $\alpha_X$  from  $x_0$  to  $g \cdot x_0$ .

**Claim:** Each vertex of  $\alpha_X$  is within  $L$  of a point of  $r$  where  $L$  depends only on  $K$  and the quasi-isometry constants coming from the natural quasi-isometry between  $G$  and  $X$  guaranteed by Theorem 2.8.

Indeed, suppose  $v \cdot x_0$  is a vertex on  $\alpha_X$ . Then  $v$  is either one of the  $g_i$ 's or  $v$  lies on  $\alpha_i$  for some  $i$ . In the first case,  $v \cdot x_0$  is within  $K$  of  $r$  by construction, in the second case,  $v \cdot x_0$  is within  $\min\{d(v \cdot x_0, g_i \cdot x_0), d(v \cdot x_0, g_{i+1} \cdot x_0)\} + K$  of  $r$ . It suffices to show that the first term of this sum is bounded by an  $L$  as in the statement of the claim.

This follows from the fact that the quasi-isometry between  $G$  and  $X$  is proper - i.e. the ball of radius  $K$  in  $X$  is mapped to a ball in  $G$  whose radius is linearly distorted by the quasi-isometry. The vertex  $v$  is on a path of the form  $g_{i+1}^{-1}g_i$  in  $G$  and all of these are contained in the preimage of the ball of radius  $2K + 1$  in  $X$  under the quasi-isometry by assumption. This gives

a bound in  $G$  on the distance  $v$  can be from the closest  $g_i$ , but now pushing back to  $X$  under the quasi-isometry gives the desired result.

Thus we now have a Cayley graph path  $\alpha_X$  from  $x_0$  to  $g \cdot x_0$  which lies in the  $L$  neighborhood of  $r$ . We straighten this path to a geodesic which  $\delta$ -tracks  $r$ . The proof is by induction on  $|S| = n$ .

Suppose there is an  $s \in S$  which occurs in  $\alpha$  and the corresponding  $s$ -wall  $W_s$  is crossed by  $\alpha$  more than once. If there is no such  $s$ , then  $\alpha$  is geodesic already. For a fixed generator  $s$ , there can be more than one  $s$ -wall which has multiple intersections with  $\alpha$ , but these can be handled independently of one another as we point out in Remark 1 below. Start by finding the first occurrence of  $s$  along  $\alpha$  so that the corresponding  $s$ -wall  $W_s$  crosses  $\alpha$  more than once. We consider the two cases of whether the number of times crossed is even or odd.

*Even case:* In this case,  $\alpha$  can be written as  $asbsc$  with the following properties:

- (1) The  $s$  edge following  $a$  is the first time  $\alpha$  crosses the  $s$ -wall  $W_s = \text{Fix}(asa^{-1})$ .
- (2) The  $s$  edge following  $b$  is the last time  $\alpha$  crosses  $W_s$ .
- (3) The  $b$  path is all on one side of  $W_s$ .
- (4) The paths  $a$  and  $c$  do not cross  $W_s$  at all.

Let  $M = \max\{d(x_0, s \cdot x_0) : s \in S\}$  and consider the  $CAT(0)$  segment  $[a \cdot x_0, asbs \cdot x_0]$ . The endpoints of this segment lie in  $N_M(\text{Fix}(asa^{-1}))$  which is a convex subset of  $X$ , therefore the entire segment lies in there. By Theorem 2.6,  $\langle lk(s) \rangle$  acts geometrically on this  $CAT(0)$  subset of  $X$ . Now the induction hypothesis applies since  $\langle lk(s) \rangle$  is a right-angled Coxeter group with no more than  $n - 1$  generators ( $s$  is not in this subgroup), that is isometrically embedded as a convex subset of  $\Lambda$  in the word metric. Thus there is a  $\delta_{n-1}$  and a Cayley graph geodesic  $\beta$  in  $\langle lk(s) \rangle$  so that  $\beta$   $\delta_{n-1}$ -tracks  $[a \cdot x_0, asbs \cdot x_0]$  and  $\beta$  is also a geodesic in  $G$ . Thus we can replace the path  $sbs$  with the geodesic  $\beta$ .

*Odd case:* In this case,  $\alpha$  can be written in the same form as above but properties 2 and 4 do not hold. In particular,  $1_G$  and  $g$  are on opposite sides of  $\text{Fix}(asa^{-1})$  and so there is exactly one more place that  $\alpha$  crosses this wall and it occurs in the  $c$  path. More specifically, the  $s$  following the  $b$  in  $\alpha$  is the second to last crossing of this wall. We do the same procedure as above to replace  $b$  with a geodesic  $\beta$  which has no more occurrences of  $s$  in it. The new path  $a\beta c$  obtained crosses  $\text{Fix}(asa^{-1})$  exactly once.

*Remark 1.* Note that the subpath  $a$  could contain the letter  $s$ , but the corresponding  $s$ -wall is different from the  $s$ -wall used above. By assumption, any  $s$ -wall that  $a$  may cross can only be crossed once. Also, the subpath  $c$  could contain  $s$  and could cross the corresponding  $s$ -wall more than once. If so, we can do the procedure described above to this subpath independent of the procedure done for  $b$  - i.e. any  $s$  deletions that occur in  $c$  will occur within  $c$  and not with any previous  $s$  occurrences by choice of  $b$ . Thus with

one use of induction, we can replace pieces of  $\alpha$  to obtain a path which has the property that any  $s$ -wall it crosses, it crosses exactly once.

*Remark 2.* Notice that every point of  $[a \cdot x_0, asbs \cdot x_0]$  is within  $L$  of  $r$  by the CAT(0) inequality.

Do this replacement procedure for each  $s \in S$  to obtain a path  $\alpha_r$  which has the property that any wall it crosses, it crosses exactly once - i.e.  $\alpha_r$  is a Cayley graph geodesic. To see  $\alpha_r$   $\delta$ -tracks  $r$ , we use the remarks above to conclude that  $\delta_n = n\delta_{n-1} + L$  suffices. Indeed, we will have to use induction at most  $n$  times, once for each  $s \in S$ .

Now suppose  $y$  is any point in  $X$ . It is clear that we can find a Cayley graph geodesic  $\alpha$  which  $(\delta = \delta_n + K)$ -tracks  $[x_0, y]$  since we can choose an orbit point  $K$  close to  $y$  and do the above procedure.

If  $r : [0, \infty) \rightarrow X$  is a geodesic ray based at  $x_0$ , then build a Cayley graph geodesic ray  $\alpha_r$  which  $\delta$ -tracks  $r$  as follows:

For each  $t \in [0, \infty)$ , we can find  $\alpha_t$  which  $\delta$ -tracks  $[x_0, r(t)]$  by the previous step. To build  $\alpha$ , use the local finiteness of  $\Lambda$  - indeed, infinitely many of the  $\alpha_t$  share the same first edge, infinitely many of these share the same second edge, etc. This clearly builds a geodesic ray with the necessary properties.  $\square$

Continuing with the quasi-isometry of the Cayley graph into the CAT(0) space for our right-angled Coxeter group  $G$  as before we give two improved versions of Proposition 4.2. The last proposition is the version we will need in the proof of the main theorem, however we need the following intermediate step to obtain that result.

**Proposition 4.3.** *Suppose  $g$  and  $h$  are at distance  $n$  apart in the Cayley graph and  $\alpha$  is a Cayley graph geodesic from  $1_G$  to  $g$ . Then there exist Cayley graph geodesics  $\alpha'$  and  $\beta'$  from  $1_G$  to  $g$  and  $h$ , respectively, such that each vertex of  $\alpha'$  and  $\beta'$  is at most  $n$  from a vertex of  $\alpha$ , and  $\alpha'$  and  $\beta'$  have the same initial segment from  $1_G$  to  $k$  for an element  $k$  with Cayley graph length  $\ell(k) \geq \ell(g) - n$ .*

*Proof.* We proceed by induction on  $n$ . If  $n = 0$  then  $g = h$  so  $\alpha' = \beta' = \alpha$  works. Suppose  $n > 0$  and take a Cayley graph geodesic  $(s_1, s_2, \dots, s_n)$  from  $g$  to  $h$ . Either  $\ell(gs_1)$  is one more than  $\ell(g)$ , or else  $\ell(gs_1)$  is one less than  $\ell(g)$  by lemma 1.5.

In the first case, denote by  $\alpha_1$  the geodesic  $(\alpha, s_1)$  from  $1$  to  $gs_1$ . Since  $h$  is at distance  $n - 1$  from  $gs_1$ , by induction hypothesis there exist  $\alpha'_1$  and  $\beta'_1$  within  $n - 1$  of  $\alpha_1$  and sharing an initial segment of length at least  $\ell(gs_1) - (n - 1)$ . Since  $(\alpha'_1, s_1)$  is not geodesic, by the Deletion Condition  $gs_1 = us_1v$ , with  $u$  and  $v$  represented by segments of  $\alpha'_1$  with  $s_1vs_1 = v$ . In fact, by lemma 1.7,  $s_1$  commutes with each letter of  $v$ . Replace the segment of  $(\alpha'_1, s_1)$  corresponding to  $s_1vs_1$  with the segment of  $\alpha'_1$  corresponding to  $v$  to give a geodesic  $\alpha'$  from  $1_G$  to  $g$  within one of  $(\alpha'_1, s_1)$ . Suppose  $k_1$  is represented by the common initial subpath of  $\alpha'_1$  and  $\beta'_1$ . If  $k_1$  is given by an initial segment in  $u$  then we simply take  $\beta' = \beta$ . Otherwise write  $k_1 = us_1w$

for  $w$  given by an initial segment of  $v$ . Then  $s_1ws_1 = w$  and we take  $\beta'$  to be the path  $\beta'_1$  with the segment corresponding to  $s_1w$  replaced by  $ws_1$ . Since  $uw$  corresponds to an initial segment of  $\alpha'$  at most one shorter than  $k_1$ , we have  $\alpha'$  and  $\beta'$  share a common initial segment of length at least  $\ell(g) - n$  and since  $\alpha'$  and  $\beta'$  are each at most 1 from  $\alpha'_1$  they are also at most one  $n$  from  $\alpha$ .

In the second case where  $gs_1$  is shorter than  $g$ , write  $g = us_1v$  for  $s_1vs_1 = v$  and  $s_1$  commuting with each letter in  $v$  as above. Take  $\alpha_1$  to be the geodesic obtained by replacing the segment of  $(\alpha, s_1)$  corresponding to  $s_1vs_1$  by  $v$ , a geodesic to  $gs_1$  with  $gs_1$  a distance at most  $n - 1$  from  $h$ . By induction hypothesis we have  $\alpha'_1$  and  $\beta'_1$  geodesics from  $1_G$  to  $gs_1$  and  $h$ , at most  $n - 1$  apart and sharing a common initial segment of length at least  $\ell(gs_1) - (n - 1) = \ell(g) - n$ . Take  $\alpha' = (\alpha'_1, s_1)$  and  $\beta' = \beta'_1$ . Then  $\alpha'$  and  $\beta'$  are at most  $n$  from  $\alpha$  and share the same common initial segment as  $\alpha'_1$  and  $\beta'_1$  of length at least  $\ell(g) - n$ .  $\square$

**Lemma 4.4.** *There exists  $c$  and  $d$  such that for any  $r$  and  $s$ , infinite geodesic rays in  $X$  based at  $x_0$ , that are within  $\epsilon$  of each other a distance  $M$  from  $x_0$ , there exist Cayley graph geodesic rays  $\alpha$  and  $\beta$  which  $(c\epsilon + d)$  track  $r$  and  $s$  respectively, and which share a common initial segment out to a distance  $M - c\epsilon - d$  from  $x_0$ .*

*Proof.* Appropriate  $c$  and  $d$  are shown below. Suppose  $r$  and  $s$  are given and take  $\alpha_0$  and  $\beta_0$  to be Cayley graph geodesics  $\delta$ -tracking  $r$  and  $s$  for a  $\delta$  from Proposition 4.2. Then at a  $CAT(0)$  space distance of no more than  $M + K$  from  $x_0$  (where  $K$  comes from the cocompactness of the action) we have orbit points  $gx_0$  and  $hx_0$  on these a Cayley graph distance  $n$  apart with  $n$  bounded by a function of  $\epsilon$  determined by the quasi-isometry constants and  $\delta_0$ . By Proposition 4.3, we can replace the segments of  $\alpha_0$  and  $\beta_0$  out to  $gx_0$  and  $hx_0$  by Cayley graph geodesic segments agreeing to a point within  $n$  of  $gx_0$  in the Cayley graph, a distance in the  $CAT(0)$  space bounded again in terms of quasi-isometry constants to be at most  $(c\epsilon + d)$  from  $gx_0$  and hence at least  $(M - c\epsilon - d)$  from  $x_0$ . Since we are replacing an initial geodesic segment of  $\alpha_0$  by a geodesic out to  $gx_0$  we again get an infinite geodesic ray  $\alpha$ , actually unchanged after  $gx_0$ , and similarly we get a  $\beta$  from  $\beta_0$ . Since  $\alpha$  and  $\alpha_0$  are within a distance  $n$  in the Cayley graph, by a constant determined from the quasi-isometry constants and  $\delta$ , the new  $\alpha$  tracks the original  $r$  within some  $(c\epsilon + d)$  (we may as well take the larger of the  $c$  and  $d$  from here and before) and similarly for  $\beta$  tracking  $s$ .  $\square$

## 5. THE FILTERING PROCESS

Recall that our ultimate goal is to start with two geodesic rays in the  $CAT(0)$  space  $X$  whose endpoints are close in  $\partial_\infty X$  and build a small connected set containing these two endpoints. From the two rays in  $X$ , we first obtain two geodesic rays  $\alpha$  and  $\beta$  in  $\Lambda$  with a long common initial segment using lemma 4.4. This is the starting point for the work in this section.

Our first goal is to use the rays  $\alpha$  and  $\beta$  to build a collection of graphs in the plane whose edges are labeled by elements of  $S$ . This collection will consist of overlapping *fans* (see definition 5.7) from which we will extract a planar, one-ended graph called a *filter* for  $\alpha$  and  $\beta$ . One should think of this process of creating a filter as trying to fill in the Cayley graph between  $\alpha$  and  $\beta$  in a systematic way so as to avoid product subgroups yet also building a sequence of geodesics between  $\alpha$  and  $\beta$ .

There is an obvious map from any such filter into  $\Lambda$  and this map into  $\Lambda$  will be proper. A continuous map  $f : X \rightarrow Y$  is *proper* if for each compact set  $C \subset Y$ ,  $f^{-1}(C)$  is compact in  $X$ . The most important property of this filter is that when viewed in  $\Lambda$ , there are no long *factor paths* (see definition 5.2 below). These are exactly the paths that behave badly under quasi-isometry so we must control the size of these paths if we hope to get any information to the  $CAT(0)$  space  $X$  from this filter. Moving this information into  $X$  will ultimately enable us to achieve our goal of constructing a “small” connected set in  $\partial_\infty X$  containing the two points, thus showing  $X$  has locally connected boundary.

Suppose  $(G, S)$  is a right-angled Coxeter system,  $G$  is a one ended group and  $G \neq \langle A \rangle \oplus \langle B \rangle$  where  $A$  and  $B$  are non-empty disjoint subsets of  $S$  such that  $A \cup B = S$ . Recall that when  $G$  is right-angled,  $lk^2(C)$  where  $C$  is a subset of  $S$  is the ordinary link,  $lk(C)$ , in  $\Gamma$  (see remarks following definition 1.10). We define the notion of *product separator* below. This should be compared with definition 3.1 from earlier in the paper as both will be needed here.

**Definition 5.1** (Product separator). *Suppose  $(G, S)$  is a Coxeter system and  $\Gamma$  is the presentation graph for  $(G, S)$ . A product separator for  $\Gamma$  is a subset  $C$  of  $S$  where  $C$  is a full subgraph,  $C = A \cup B$ ,  $\langle C \rangle = \langle A \cup B \rangle = \langle A \rangle \oplus \langle B \rangle$ , both  $\langle A \rangle$  and  $\langle B \rangle$  are infinite and  $C$  separates  $\Gamma$ .*

**Definition 5.2** (Factor path). *Call a path  $(c_1, \dots, c_m)$  in  $\Lambda$  a factor path if  $\{c_1, \dots, c_m\} \subset A \cup B$  where  $A$  and  $B$  are disjoint commuting subsets of  $S$  and  $\langle A \rangle$  and  $\langle B \rangle$  are infinite.*

**Remark** Suppose  $(G, S)$  is a Coxeter system with presentation graph  $\Gamma$ .  $G$  is one-ended if and only if  $\Gamma$  contains no complete separating subgraph, the vertices of which generate a finite subgroup. A proof of this can be found in [MT].

The following lemma allows us to easily identify when the presentation graph  $\Gamma$  for  $G$  has a virtual factor separator given that there are no product separators. This lemma will be used repeatedly in lemmas 5.5 and 5.6 below.

**Lemma 5.3.** *Suppose  $\Gamma$  has no product separators. Then  $\Gamma$  has a virtual factor separator iff there is a vertex  $x \in \Gamma$  such that  $lk(x)$  is a subset of a visual product with infinite factors.*

*Proof.* Suppose  $\langle A \rangle \oplus \langle B \rangle$  is a visual product in  $(G, S)$  with infinite factors,  $\langle B \rangle$  has finite index in  $\langle B' \rangle$  and  $B'$  separates  $\Gamma$  (i.e.  $(B', B, A)$  is a virtual factor separator).

By the Finite Index Lemma (lemma 1.16), there exists  $B_1 \subset B$  such that  $\langle B' - B_1 \rangle$  is finite and commutes with  $B_1$ . Hence  $\langle B' \rangle = \langle B_1 \rangle \oplus \langle B' - B_1 \rangle$ .

Note that  $A \cup B' \neq S$ , as  $A \cup (B' - B_1)$  commutes with  $B_1$  and  $G$  does not decompose as a non-trivial visual direct product. Let  $y$  be a vertex in  $\Gamma - (A \cup B')$  and  $x$  be a vertex of  $\Gamma$  separated from  $y$  by  $B'$ . We have  $x \in A$ , for otherwise,  $\langle A \cup B' \rangle = \langle A \cup (B' - B_1) \rangle \oplus \langle B_1 \rangle$  is a product separator for  $\Gamma$ . If  $c \in lk(x)$ , and  $c \notin B'$ , then  $x$  and  $c$  are in the same component of  $\Gamma - B'$ . Hence  $B'$  separates  $c$  and  $y$  and as above  $c \in A$ . This means that  $lk(x) \subset A \cup B' \subset \langle A \cup (B' - B) \rangle \oplus \langle B_1 \rangle$ .

To see the converse, suppose  $x$  is a vertex of  $\Gamma$  and  $lk(x) \subset (A \cup B) \subset S$  where  $A$  commutes with  $B$  and both  $\langle A \rangle$  and  $\langle B \rangle$  are infinite. The group  $\langle lk(x) \rangle$  is infinite as  $G$  has one end. If both  $\langle lk(x) \cap A \rangle$  and  $\langle lk(x) \cap B \rangle$  were infinite, then  $\Gamma$  has a product separator. If one, say  $\langle lk(x) \cap A \rangle$ , is finite, then  $\langle lk(x) \cap B \rangle$  commutes with  $A$  and so  $(lk(x), lk(x) \cap B, A)$  is a virtual factor separator.  $\square$

**Remark.** For the rest of this section, in addition to the basic assumptions at the beginning of the section, assume  $\Gamma$  has no product separator and no virtual factor separator.

*Notation.* The vertices of  $\Gamma$  are the elements of  $S$  and each edge of  $\Lambda$  is labeled by an element of  $S$ . If  $e$  is an edge in  $\Lambda$ , then we let  $\bar{e} \in S \subset G$  denote the label of  $e$ . Denote the vertex sets of  $\Gamma$  and  $\Lambda$  by  $\Gamma^0$  and  $\Lambda^0$ , respectively. Also, recall that  $B(g)$  is the set of elements of  $S$  that make  $gs$  shorter than  $g$  and that this generates a finite subgroup (lemma 1.5).

To build a fan, we begin with edge paths  $(e_1, \dots, e_n, a)$  and  $(e_1, \dots, e_n, b)$  that are geodesic in  $\Lambda$ , and let  $g \equiv \bar{e}_1 \cdots \bar{e}_n \in G$ . Suppose  $(e_i, \dots, e_n)$  is the longest (terminal) factor path in  $(e_1, \dots, e_n)$ . This may in fact be a trivial path, as in the case  $G$  is word hyperbolic.

If  $\{\bar{e}_i, \dots, \bar{e}_n\}$  is infinite and  $A$  and  $B$  are as in the definition of factor path, then  $\langle \{\bar{e}_i, \dots, \bar{e}_n\} \cap A \rangle$  or  $\langle \{\bar{e}_i, \dots, \bar{e}_n\} \cap B \rangle$  is infinite. Without loss, assume  $C \equiv \{\bar{e}_i, \dots, \bar{e}_n\} \cap A$  and  $\langle C \rangle$  is infinite. Now  $\{\bar{e}_i, \dots, \bar{e}_n\} \subset C \cup lk(C)$ . Also  $B \subset lk(C)$  so  $\langle C \rangle$  and  $\langle lk(C) \rangle$  are infinite.

**Lemma 5.4.** *If  $\{\bar{e}_i, \dots, \bar{e}_n\}$  is infinite, then  $\{\bar{e}_i, \dots, \bar{e}_n\} \cup B(g) \subset C \cup lk(C)$ . Hence  $\langle C \rangle \oplus \langle lk(C) \rangle$  is a visual direct product with both factors infinite and so  $C \cup lk(C)$  does not separate  $\Gamma$ .*

*Proof.* Recall,  $s \in S$  is in  $B(g)$  if  $gs$  is shorter than  $g$ . If  $s \in B(g)$ , then let  $j$  be the largest integer such that for the path  $(e_1, \dots, e_n, s)$ ,  $s$  deletes with  $e_j$ . If  $j \geq i$ , then  $s \in \{\bar{e}_i, \dots, \bar{e}_n\} \subset C \cup lk(C)$ . If  $j < i$ , then by lemma 1.7,  $s \in lk(C)$ . In any case  $B(g) \subset C \cup lk(C)$ .  $\square$

**Important:** The next two lemmas will allow us to choose a path in  $\Gamma$  from  $a$  to  $b$  that avoids certain subsets of  $\Gamma$ . We use this path to build a fan

between these two edge paths. We split this into two lemmas depending on whether  $\langle \bar{e}_i, \dots, \bar{e}_n \rangle$  is finite or infinite. Note that we allow the case  $a = b$ .

**Lemma 5.5.** *If  $\langle \bar{e}_i, \dots, \bar{e}_n \rangle$  is infinite, then there is an edge path  $\tau$  in  $\Gamma$ , from  $a$  to  $b$  of length at least 2, such that other than possibly  $a$  and  $b$ , no vertex of this path is in  $C \cup lk(C)$ .*

*Proof. Case 1.* Suppose  $a = b$ .

By Lemma 5.3, there is a vertex  $v \in lk(a)$  such that  $v \notin C \cup lk(C)$ . If  $e$  is the edge of  $\Gamma$  from  $a$  to  $v$  then our path is  $(e, e^{-1})$ .

**Case 2.** Suppose  $a \in C \cup lk(C)$  and  $b \notin C \cup lk(C)$  (or when the roles of  $a$  and  $b$  are reversed).

By Lemma 5.3, there is an edge from  $a$  to  $v \notin C \cup lk(C)$ . If  $v \neq b$ , then (as  $C \cup lk(C)$  cannot separate  $\Gamma$ ) take as the desired path, the edge from  $a$  to  $v$  followed by any edge path from  $v$  to  $b$  in  $\Gamma - (C \cup lk(C))$ . If  $v = b$ , then by Lemma 5.3 there is a vertex  $w \in lk(v)$  such that  $w \notin C \cup lk(C)$ . If  $e$  is the edge of  $\Gamma$  from  $v$  to  $w$ , then take the desired path to be the edge from  $a$  to  $v$  followed by  $e$ , followed by  $e^{-1}$ .

**Case 3.** Suppose  $a, b \in C \cup lk(C)$ .

By Lemma 5.3, there is an edge  $e$  from  $a$  to  $v \notin C \cup lk(C)$  and an edge  $d$  from  $b$  to  $w \notin C \cup lk(C)$ . Take as the desired path, the edge  $e$  followed by a path in  $\Gamma - (C \cup lk(C))$  from  $v$  to  $w$  followed by  $d^{-1}$ .

**Case 4.** Suppose  $a, b \notin C \cup lk(C)$  and  $a \neq b$ .

Choose as desired path, any path of length at least 2 in  $\Gamma - (C \cup lk(C))$  from  $a$  to  $b$ . (Again, if there is an edge between  $a$  and  $b$  one can adjust as in Case 2.)  $\square$

**Remark.** It seems somewhat artificial to consider an edge followed by its inverse in our above paths, but paths of length at least 2 in length induce beneficial combinatorics in our constructions. These combinatorics allow for a simplified proof of lemma 5.8 at the end of the section.

**Lemma 5.6.** *If  $\langle \bar{e}_i, \dots, \bar{e}_n \rangle$  is finite, there is a path  $\tau$ , in  $\Gamma$  from  $a$  to  $b$ , of length at least 2 which avoids  $B(g)$ .*

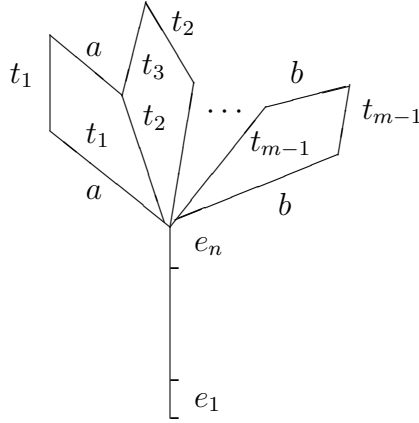
*Proof.* Recall that  $B(g)$  cannot separate  $\Gamma$  as  $G$  is 1-ended (see remark preceding lemma 5.3). Now proceed as in lemma 5.5, case 4.  $\square$

Given any edge paths  $(e_1, \dots, e_n, a)$  and  $(e_1, \dots, e_n, b)$ , we now have the path  $\tau$  in  $\Gamma$  from  $a$  to  $b$  which avoids the appropriate subsets of  $\Gamma$  depending on whether  $\langle \bar{e}_i, \dots, \bar{e}_n \rangle$  is finite or infinite. Let the ordered vertices of  $\tau$  be  $a = t_0, t_1, \dots, t_m = b$  ( $m \geq 2$ ). Note that in  $G$ ,  $[t_i, t_{i+1}] = 1$  for all  $i$ .

We now describe the construction of a *fan* using the path  $\tau$ . These fans will be the buliding blocks in the construction of a *filter* for two geodesic rays  $\alpha$  and  $\beta$  in  $\Lambda$ .

**Definition 5.7 (Fan).** *The  $\tau$ -fan for  $(e_1, \dots, e_n, a)$  and  $(e_1, \dots, e_n, b)$  is a planar diagram consisting of edges labeled  $e_1, \dots, e_n, a, b$  plus loops labeled by the vertices of the path  $\tau$  defined in Lemma 5.5 or 5.6 (see Figure 1*

below). The fan-loops are four sided loops with edge labels  $(t_i, t_{i+1}, t_i, t_{i+1})$  beginning at a vertex labeled  $g \equiv \bar{e}_1 \cdots \bar{e}_n$ . These loops correspond to the commutation relations  $[t_i, t_{i+1}]$  among consecutive vertices of  $\tau$ . We will simply refer to this as a fan for  $(e_1, \dots, e_n, a)$  and  $(e_1, \dots, e_n, b)$  if the path  $\tau$  is unimportant.

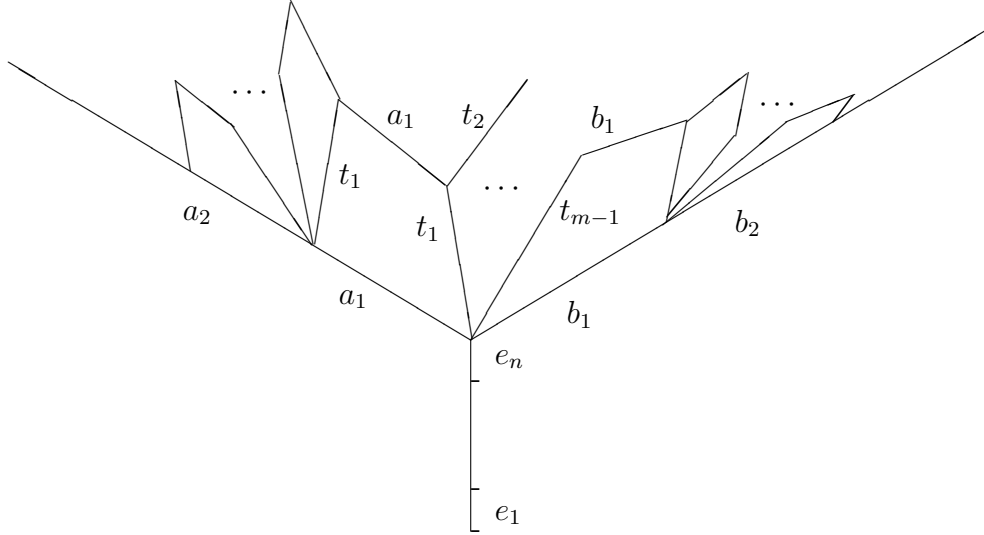


**Figure 1**

Recall that  $t_i \notin B(g)$  for all  $i$ . Hence, by lemma 1.4, the edge paths  $(e_1, \dots, e_n, t_i, t_{i+1})$  in  $\Lambda$  are geodesic. The edges labeled  $a$  and  $b$  at  $g$  in the  $\tau$ -fan are respectively called the *left* and *right fan edges* at  $g$ . The edges labeled  $t_1, \dots, t_{m-1}$  at  $g$  are called *interior fan edges*.

Next, suppose  $\alpha = (e_1, \dots, e_n, a_1, a_2, \dots)$  and  $\beta = (e_1, \dots, e_n, b_1, b_2, \dots)$  are geodesics in  $\Lambda$ .

**The Filter Construction:** Construct a 1-ended planar graph (see Figure 2 below) with edge labels in  $S$  as follows: First construct a fan for  $(e_1, \dots, e_n, a_1)$  and  $(e_1, \dots, e_n, b_1)$ . Next overlap this fan with a fan for  $(e_1, \dots, e_n, a_1, a_2)$  and  $(e_1, \dots, e_n, a_1, t_1)$  a fan for  $(e_1, \dots, e_n, t_1, a_1)$  and  $(e_1, \dots, e_n, t_1, t_2), \dots$ , a fan for  $(e_1, \dots, e_n, t_i, t_{i-1})$  and  $(e_1, \dots, e_n, t_i, t_{i+1}), \dots$  and finally a fan for  $(e_1, \dots, e_n, b_1, t_{m-1})$  and  $(e_1, \dots, e_n, b_1, b_2)$ .


**Figure 2**

This gives our 1<sup>st</sup> and 2<sup>nd</sup>-level fans.

At this point there is the potential for two edges of our planar graph to share a vertex and have the same label. These edges are *not* identified in our planar graph. Rather, an edge path  $\tau$ , constructed as in case 1 of lemma 5.5 will be used to extend our graph between these two edges.

To continue, we must specify geodesics from  $*$  to each vertex defined so far. In each fan-loop constructed up to this point, designate the upper left edge as a *non-tree* edge. The graph minus the non-tree edges is a tree. Take as designated geodesic from  $*$  to a defined vertex, the unique geodesic of the tree.

Continuing, (at each stage designating upper left edges of fan loops as non-tree edges and constructing fans) we obtain the desired 1-ended planar graph called a *filter* for  $\alpha$  and  $\beta$ . Geodesics from  $*$  in this filter determine geodesics in  $\Lambda$  with the same edge labels. Hence there is a natural proper map from this filter to  $\Lambda$ . The image of this filter in  $\Lambda$  is a closed 1-ended subgraph.

The following are useful combinatorial facts about a filter  $F$ .

- (1) Each vertex of  $F$  has exactly one or two edges directly below it. This fact would be missing if certain identifications had been allowed.
- (2) If a vertex of  $F$  has exactly one edge below it, then this edge is either an interior fan edge, an  $\alpha$ -edge or a  $\beta$ -edge.
- (3) If a vertex of  $F$  has exactly two edges below it, then one of these edges is a left fan edge and the other is a right fan edge.
- (4) The filter for  $\alpha$  and  $\beta$  minus all non-tree edges is a tree containing  $\alpha$  and  $\beta$  and all interior edges of all fans.

- (5) An edge of the filter is a non-tree edge iff it is a right fan edge not on  $\beta$ .
- (6) Let  $T$  be the tree obtained from a filter  $F$  by removing all non-tree edges. There are no dead ends in  $T$  - i.e. for any vertex  $v \in T$ , there is an interior edge at  $v$ .

We now begin the process of showing that if we create a filter for geodesic rays  $\alpha$  and  $\beta$  that share a long common initial piece, then there is a bound on the length of any factor path in this filter. Again, this will allow us to map the one-ended planar graph into  $X$  and obtain a small connected set in  $\partial_\infty X$ . The final result necessary is lemma 5.10 but we prove two intermediate lemma about factor paths in filters along the way.

**Lemma 5.8.** *Let  $\gamma \equiv (c_1, \dots, c_n)$  be an edge path in  $T$  such that each edge is above the previous edge and at most, the initial point of  $\gamma$  intersects  $\alpha \cup \beta$ . Suppose  $(c_i, \dots, c_j)$  is a factor path of  $\gamma$  of length  $\geq |S|$ . Then, either  $\bar{c}_{j+1}$  or  $\bar{c}_{j+2}$  is not in  $\{\bar{c}_i, \dots, \bar{c}_j\}$ .*

*Proof.* Neither,  $c_{j+1}$  nor  $c_{j+2}$  is a right fan edge, as both are in  $T$ . If  $c_{j+1}$  is an interior fan edge we are done by the construction in lemma 5.5. Hence we may assume that  $c_{j+1}$  is a left fan edge and  $c_{j+1} \in \{c_i, \dots, c_j\}$ . Now, if  $c_{j+2}$  is an interior fan edge, once again we are finished. Hence we may assume  $c_{j+2}$  is a left fan edge. In fact, we argue that this last situation is impossible.

Let  $u$  be the end point of  $c_{j+2}$ . As  $c_{j+2}$  is a left fan edge, say  $a$  is the right fan edge ending at  $u$ .

If  $\bar{a} = \bar{c}_{j+1}$  then  $c_{j+1}$  must be an interior or right fan edge (at  $w$ ), contrary to our assumption. Hence the situation must be one of the following two cases.

**Case 1.** Suppose  $\bar{a} \notin \{\bar{c}_i, \dots, \bar{c}_j\}$ .

Consider the path  $(c_i, \dots, c_{j+2})$  at  $z$ . The edge  $c_i$  at  $z$  is either an internal or left fan edge and all other edges of this path are internal fan edges. Hence this entire path is in  $T$ . But then by the Lemma 3 construction,  $c_{j+1} \notin \{c_i, \dots, c_j\}$ , contrary to our assumption. •

**Case 2.** Suppose  $\bar{a} \in \{\bar{c}_i, \dots, \bar{c}_j\}$ .

Let  $k$  be the largest integer in  $\{i, \dots, j\}$  such that  $\bar{a} = \bar{c}_k$ . Note that  $\bar{a} \neq \bar{c}_{j+1}$  as this was considered above. The edge path  $(c_i, \dots, c_{k-1}, c_{k+1}, \dots, c_{j+2})$  at  $x$  is in  $T$ , as each edge of the subsegment  $(c_{k+1}, \dots, c_{j+2})$  is either an internal or left fan edge. Hence  $(c_i, \dots, c_{k-1}, c_{k+1}, \dots, c_{j+2})$  at  $x$  defines a (geodesic) factor path. As the length of this geodesic is  $\geq |M| + 2$ ,  $\langle \bar{c}_i, \dots, \bar{c}_{k-1}, \bar{c}_{k+1}, \dots, \bar{c}_j \rangle$  is infinite. The  $c_k$ -edge at  $y$  is a right fan edge and so  $c_{j+1}$  at  $y$  is an interior fan edge at  $y$ . Hence our  $\tau$ -fan construction in lemma 5.5 implies that  $\bar{c}_{j+1} \notin \{\bar{c}_i, \dots, \bar{c}_{k-1}, \bar{c}_{k+1}, \dots, \bar{c}_j\}$ . But  $\bar{c}_{j+1} \neq \bar{c}_k$  as noted earlier. This is contrary to our assumption.  $\square$

**Lemma 5.9.** *Suppose that  $\gamma \equiv (c_1, \dots, c_n)$  is a directed factor path in  $T$  such that  $\gamma$  intersects  $\alpha \cup \beta$  in at most, its initial point. Then  $n \leq 3|S|$ .*

*Proof* By lemma 5.8, at least every other edge  $e_i$  following  $e_{|S|}$  is such that  $\bar{e}_i \notin \{\bar{e}_1, \dots, \bar{e}_{i-1}\}$ .  $\square$

The following lemma is the main lemma of this section that will be used in the proof of main theorem. In order to obtain the necessary bounds on the lengths of factor paths, we must analyze how such a path arises in  $\Lambda$ . This involves studying Van-Kampen diagrams in the group. We refer the reader to [LS] for the necessary background concerning Van-Kampen diagrams for groups (abbreviated V-K diagrams).

**Lemma 5.10.** *For any  $k$ , there exists a  $\delta$  such that, for any  $\alpha$  and  $\beta$  Cayley graph geodesics between the same pair of points, if for some  $n$ ,  $\alpha(n)$  and  $\beta(n)$  are at least  $\delta$  apart, then  $\alpha$  contains a factor subpath of length  $k$  (and also  $\beta$  has such a subword).*

*Proof.* Write  $\alpha = (\alpha_0, \alpha_1)$  and  $\beta = (\beta_0, \beta_1)$  for  $\alpha_0$  and  $\beta_0$  initial subpaths of length  $n$ . Take a Cayley graph geodesic  $\gamma$  from  $\alpha(n)$  to  $\beta(n)$  and V-K diagrams for  $\alpha_0\gamma = \beta_0$  and  $\alpha_1 = \gamma\beta_1$  each with a minimum number of regions, such that the combination is a V-K diagram for the relation telling us that  $\alpha$  and  $\beta$  represent the same element of the Coxeter group. Given any edge in this V-K diagram, we construct a chain of boxes corresponding to commuting generators in the defining presentation. Such a chain cannot cross  $\gamma$  twice or begin and end on  $\alpha$  or begin or end on  $\beta$  since each of these is a geodesic. Such a chain cannot be a loop else one of the diagrams wouldn't have had a minimum number of regions. No two chains with the same letter cross each other. Thus each chain begins on  $\alpha$ , ends on  $\beta$ , and perhaps crosses  $\gamma$  with as many chains crossing from  $\alpha_0$  to  $\beta_1$  as from  $\alpha_1$  to  $\beta_0$ . Note that  $\gamma$  consists of an edge from each such chain. Each letter for a chain from  $\alpha_0$  to  $\beta_1$  commutes with a letter on a chain from  $\alpha_1$  to  $\beta_0$  (since these chains cross at some point) which gives a commutation relation between these letters. Write  $\gamma_0$  for the subsequence of letters of  $\gamma$  that label chains extending from  $\alpha_0$  to  $\beta_1$  and  $\gamma_1$  for the subsequence of letters of  $\gamma$  that label chain extending from  $\alpha_1$  to  $\beta_0$ . Then in fact  $\gamma$  and  $(\gamma_0, \gamma_1)$  would represent the same element since each letter of  $\gamma_0$  commutes with each letter of  $\gamma_1$ .

We take  $\delta$  greater than twice the longest geodesic in a finite subgroup of the Coxeter group. Then the letters in  $\gamma_0$  generate an infinite subgroup as do the letters in  $\gamma_1$ . Let  $\alpha'_0$  be the subsequence of letters in  $\alpha_0$  belonging to chains crossing  $\gamma$  (a word equivalent to  $\gamma_0^{-1}$ ). A letter of  $\alpha_0$  not in  $\alpha'_0$  must commute with each earlier letter belonging to  $\alpha'_0$  since the chain extending from such a letter extends to  $\beta_0$  and crosses the chains of earlier letters in  $\alpha'_0$ . Of the  $\delta/2$  letters in  $\alpha'_0$ , one letter must occur at least  $\delta/(2|S|)$  times say  $s_1$ . Between occurrences of  $s_1$  there must be a letter not commuting with  $s_1$  and hence also a letter of  $\alpha'_0$ , and among all such letters between successive  $s_1$

occurrences, there must be some letter occurring at least  $(\delta/(2|S|) - 1)/|S|$  times say  $s_2$ . We define a subset  $S'$  of the letters  $s_1, s_2, \dots, s_i$  occurring in  $\alpha'_0$ , and define a sequence of subwords of  $\alpha'_0$  as follows.

First we take  $s_1$  and  $s_2$  in  $S'$  and take the terminal subword of  $\alpha'_0$  after the later of the first occurrences of  $s_1$  and  $s_2$ . Now the first occurrence of  $s_3$  occurs either in the first half or the last half of this subword of  $\alpha'_0$ . If the terminal segment of  $\alpha'_0$  after the first occurrence of  $s_3$  in the remaining word is longer than the word before this  $s_3$ , then we keep  $s_3$  in  $S'$  and take the terminal segment after this  $s_3$ , and otherwise we omit  $s_3$  from  $S'$  and restrict to the  $s_3$ -free initial subword of the remainder. Repeat the process for  $s_4, \dots, s_i$  - each time taking the first occurrence of the next letter and dividing the remaining word at this point keeping the larger half, putting the letter in  $S'$  if we take a terminal segment, and otherwise omitting it from  $S'$ .

In the end we are left with a subword  $\alpha''_0$  of  $\alpha'_0$  of length at least  $2(\delta - 2|S|)/(|S|^2 2|S|)$  such that the following conditions hold: each letter in the  $\alpha''_0$  belongs to  $S'$  and each letter in  $S'$  occurs either at the beginning of  $\alpha''_0$  or else earlier in  $\alpha'_0$  than all of  $\alpha''_0$  as well. Let  $\phi$  be smallest subword of  $\alpha_0$  containing the corresponding letters in  $\alpha''_0$ . Then the letters in  $\phi$  belong to  $S'$  or belong to the subset  $lk(S')$  of letters commuting with all elements of  $S'$ . Since the letters in  $\gamma_1$  are in  $lk(S')$ ,  $lk(S')$  generates an infinite subgroup, as does  $S'$  since  $s_0$  and  $s_1$  are in  $S'$ . Hence there is a subword in  $\alpha_0$  with letters in a product of infinite subgroups having length bounded below by a function of  $\delta$ . If  $\delta \geq |S|^2 2|S|k + 2|S|$  then this subword will be a factor path of length at least  $k$ .

## 6. PROOF OF THE MAIN THEOREM

**Theorem 6.1.** *Suppose  $(G, S)$  is a right-angled Coxeter system,  $G$  is 1-ended,  $\Gamma(S)$  contains no product separator and no virtual factor separator. Also assume that  $G$  does not visually split as a non-trivial direct product. Then  $G$  has locally connected boundary.*

*Proof.* Suppose  $G$  acts geometrically on a  $CAT(0)$  space  $X$ . Choose a base point  $* \in X$ . Suppose  $r$  and  $s : [0, \infty) \rightarrow X$  are geodesics at  $*$  and  $s \in N_{(M, \epsilon)}(r)$  (i.e.  $d_X(r(M), s(M)) < \epsilon$ ).

Choose geodesics  $\alpha = (e_1, \dots, e_n, a_1, a_2, \dots)$  and  $\beta = (e_1, \dots, e_n, b_1, b_2, \dots)$  from  $\Lambda$  that  $\delta$  track  $r$  and  $s$  respectively as in lemma 4.4. Recall  $\alpha_X$  means view the path  $\alpha$  inside  $X$ . Thus we can choose  $\alpha$  and  $\beta$  so that  $\alpha_X$  and  $\beta_X$  share a common initial piece that is  $M'$  long where  $M'$  comes from following the  $M - c\epsilon - d$  from lemma 4.4 under the quasi-isometry between  $\Lambda$  and  $X$ . Note that  $d(\alpha_X(M'), r(M')) < \delta$  and  $d(\beta_X(M'), r(M')) < \delta$ .

Construct a filter  $F$  for this pair of edge paths and consider the tree  $T$  constructed from  $F$  as in section 5. Let  $f : F \rightarrow X$  be the natural map that factors through  $\Lambda(G, S)$ . To show local connectivity at  $r$ , it suffices to show

the limit set of  $f(F)$  is a “small” connected subset of  $\partial X$  containing both  $r$  and  $s$ .

The map  $f$  is proper and  $F$  is one-ended, thus the limit set of  $f(F)$  is connected. It is clear that this limit set contains  $r$  and  $s$ . To see that this limit set is small, we must apply lemma 5.10 and lemma 5.9. Here is the important point.

*Claim:* There exists  $M'$  and  $\epsilon'$  such that  $f(F) \subset N_{(M',\epsilon')}(r)$  (the  $M'$  mentioned above is the same  $M'$  used here).

First we work in  $F$ , then transfer this information over to  $X$  via map  $f$ .

Let  $v$  be a vertex in  $F$  that is far away from  $\alpha(n)$ . In particular, we want  $d(*, f(v))$  to be much bigger than  $M'$  since we are concerned with the limit set of  $F$ . Also, denote by  $p$  the point at which  $\alpha$  and  $\beta$  start to separate.

We know the geodesic in  $F$  from the identity to  $v$  is the obtained by following  $\alpha$  (and  $\beta$ ) to  $p$ , then using the unique geodesic in the tree  $T$  (described in section 5) from  $p$  to  $v$ . Call this geodesic  $\gamma$ . Thus  $\gamma(n) = \alpha(n) = \beta(n) = p$ .

By lemma 5.9 there exists a global bound on the length of a (directed) factor path in  $T$  that only intersects  $\alpha \cup \beta$  in it's initial point, call this bound  $k$ . By lemma 5.10, there is a  $\delta_1$  coming from this  $k$  such that all geodesics from the identity to  $v$  are contained in the  $\delta_1$  neighborhood of  $\gamma$ .

We transfer all of this information over to  $X$  via the quasi-isometry to obtain new constants. In particular, the  $CAT(0)$  geodesic,  $\tau$  from  $*$  to  $f(v)$  is within  $\delta_2$  of  $f(\gamma)$ . Thus  $d(\tau(M'), f(p)) < \delta_2$ .

We know from the choice of  $\alpha$  and  $\beta$  above that  $d(\alpha_X(M'), r(M')) < \delta$  and  $d(\beta_X(M'), r(M')) < \delta$ . But  $\alpha_X(M')$  is simply  $f(p)$  here, thus we have  $d(f(p), r(M')) < \delta$ . Putting these together with the triangle inequality yields:  $d(r(M'), \tau(M')) < \delta_2 + \delta$ . If we use the  $M'$  mentioned above along with  $\epsilon' = \delta_2 + \delta$ , we have the claim.  $\square$

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