

Answer Key
Hw 1

1. Define

$$A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Then the result is $A_5 A_3 A_2 B A_1 A_4 A_6 A_7$

(b) We can write this as ABC if we let $A = A_5 A_3 A_2$ and $C = A_1 A_4 A_6 A_7$

2. There are 3 equations: $2 = m * 1 + b$, $3.2 = m * 2 + b$, $5.0 = m * 4 + b$.

(a) In matrix-vector form these would be

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 2 \\ 3.2 \\ 5.0 \end{bmatrix}.$$

(b) There are only 2 columns in A . They are linearly independent because one cannot be written as a linear combination (i.e. as a multiple) of the other.

(c) There is no exact solution to this equation. You can argue this in multiple ways. First, it takes only 2 distinct points to define a line through those two points. If you take any pair of these 3 points, it's clear that the 3rd point is not on that line. If you form the augmented matrix $[A|r]$ and do row reduction on it, you find that the last row corresponds to the equation $0 = 1$, which is not true, so there's no solution. Note that the right hand side vector is not in the column space of A – it is not a linear combination of the columns of A , since you couldn't find a solution. Note also that the fact that the two columns of A are independent (a "good" thing) doesn't mean you will have a solution.

3. (This answers both parts of the question) Since C is not invertible, its null space is non-trivial – that is, there must be some non-zero vector, call it z , such that $Cz = 0$. That means $BCz = B(Cz) = B * 0 = 0$. But now we have found a non-zero vector z such that $Az = 0$, so that means A has a non-trivial null space. Therefore, A cannot be invertible, which means its columns cannot span the whole space \mathbb{R}^m . That means there are some vectors b that we can construct that are in \mathbb{R}^m but not in the column space of A (i.e. vectors which cannot be written as a linear combination of the columns of A). But in this case, there may be *no* solution.

4. A basis for \mathbb{R}^m must contain exactly m , linearly independent vectors of length m . Since $m < n$, the number of columns of A exceeds m , and therefore they cannot form an independent set. So the answer is no.

5. Note $A = \sqrt{2}/2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Ax takes x and maps it to the line through the point $(1,0)$ (a clockwise rotation of $\pi/4$). In particular, since all vectors on the line at 45 degrees are of the form $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$, $A \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \sqrt{2x_1^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \|x\|_2^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. which means that the length of x is preserved.

(b) Similarly, if x is on the line through $(-1,1)$, Ax maps to a point on the line through $(0,1)$ (still a clockwise rotation of $\pi/4$.) In particular, $A \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \sqrt{2x_1^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \|x\|_2^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Because these two vectors (I'll call them v_1, v_2) are independent, they form a basis for \mathbb{R}^2 . So any vector x can be written $x = \alpha_1 v_1 + \alpha_2 v_2$. So $Ax = \alpha_1 A v_1 + \alpha_2 A v_2 = \alpha_1 \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \sqrt{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since we identified the action on v_1, v_2 in the first 2 problems as a clockwise rotation of $\pi/4$, it follows that Ax for any x is a clockwise rotation of $\pi/4$. (It is also possible to prove that the length of x is preserved, but we'll talk about that later in the term.)

(d) Recall that a matrix-matrix product AB can be computed as $[Ab_1, Ab_2, \dots, Ab_n]$. From the first 2 parts, we have

$$A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} = \sqrt{2}I$$

So, $A(1/\sqrt{2}) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = I$, which means the inverse of A is

$$A^{-1} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

6. First, we consider the cost of a single outer product. Since a_i has m entries and \tilde{b}_i^T has p entries, and forming the (k, l) entry in the outer product only requires multiplying the k th entry in a_i with the l th entry in \tilde{b}_i^T , there will be mp multiplies to form the outer product. Since we need to compute n such outer products, there are nmp multiplies. We also have to add each of the $m \times p$ matrices together to find the sum. There are $n - 1$ matrix adds, and we have to add entries point-wise (total of mp pointwise adds), for a total of $(n - 1) * (mp)$ adds. So total flops is $nmp + (n - 1)mp$ - in big-Oh, this is $O(nmp)$ flops.