1. Text 24.1 b-g.

   b) False. Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, which obviously has 2 eigenvalues $4, 1$.

   c). True. If $Ax = \lambda x$, then $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$ (conjugating both sides), and since A is real, $A = \bar{A}$, and we're done.

   d.) True. If $Ax = \lambda x$ and A is invertible, then multiplying both sides by A^{-1} and dividing by λ (note λ cannot be zero since A is invertible)

$$\frac{1}{\lambda}x = A^{-1}x$$

   so $\frac{1}{\lambda}$

   e.) False. Any **strictly** lower or upper triangular matrix has all zero eigenvalues but is not the zero matrix. For example,

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

   f.) True. If A is Hermitian, then A is unitarily diagonalizable, so $A = V\Lambda V^*$, and we know the eignvalues are real. Let P be a permutation matrix such that $P\Lambda P^T$ has its diagonal entries ordered largest to smallest in magnitude. Let Q be a diagonal matrix with -1 if λ_i is negative and 1 if it's nonnegative or zero. Clearly Q is unitary, and so is P. Also $\Lambda = \tilde{\Lambda}Q$, where $\tilde{\Lambda}$ has $|\lambda_i|$ on its diagonal.

$$A = V \underbrace{\tilde{\Lambda}Q}_{\Lambda} V^* = \underbrace{VP^T}_{U} \underbrace{P\tilde{\Lambda}QP^T}_{\Sigma} \underbrace{PV^*}_{W^*}.$$

   This is clearly an SVD of A. So the $|\lambda_i|$ are the singular values of A.

   g.) True. If A is diagonalizable, then there exists invertible X such that

$$A = X\Lambda X^{-1}$$

   But if $\Lambda = \mu I$ (that is, all the eigenvalues are equal to μ) then

$$A = X(\mu I)X^{-1} = \mu XX^{-1} = \mu I$$

   so A is diagonal.

2. Text 25.3. For matrix (a), the answer is ii. The reason it cannot be (i) is that once you introduce those zeros below the main diagonal, if you try to do a left multiply to put a zero in the (1,3) position (which can be done by a plane rotation – just consider the 2-vector $x = [a_{1,3}; a_{2,3}]$ and rotate to the point $[0; \sqrt{a_{1,3}^2 + a_{2,3}^2}]$ – then $Q = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ does the trick

), Q *also* affects the $2,1$ position, and it will fill back in to something non-zero.

The reason it can be (ii): First, introduce zeros under the (1,1) position by your favorite method (Householder or Givens rotations). Let's call the resulting matrix $\hat{A}$ (so $\hat{A} = Q_1 A$). Then, to get the zero in the (1,3) position, define $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & s & -c \end{bmatrix}$, where

$$x = [\hat{a}_{1,2}, \hat{a}_{1,3}]; c = x(1)/\|x\|; s = x(2)/\|x\|$$

and right multiply by Q_2. Note that doing the left multiply by Q_2 does not change the first row, so the zeros are preserved. Now we have $Q_1 A Q_2$ Finally, we can introduce a zero into the (3,2) position of $\tilde{A}$ using a Givens rotation Q_2 – since Q_2 has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

then Q_3 doesn't destroy any of the zeros that are already in the matrix.

In the end we have $Q_3 Q_1 A Q_2$ with the desired form.

3. It should be obvious from your experiments that the larger the relative gap between λ_1 and λ_2 (i.e. the larger the value of $|\lambda_2|/|\lambda_1|$, the faster the convergence of the method. If you stop iterating too soon, and the gap is not large, the estimates are not as good.