Lecture Notes 3, Math/Comp 128, Math 250

Misha Kilmer
Tufts University

September 23, 2007

Projectors

Projector: A *square matrix* P that satisfies

$$P^2 = P.$$

(Any matrix such that $P^2 = P$ is called idempotent.)

Projectors

$$P^2 = P$$

Two classes of projectors:

- Orthogonal
- Nonorthogonal (Oblique)

Visual Interpretation

Orthogonal projections are easy to visualize if we collapse the picture to 2 dimensions:

Let $\mathcal{R}(P)$ be denoted by a line through the origin. Given vector v, Pv is the vector from the origin that is the shadow of v in $\mathcal{R}(P)$ caused when we shine a light on $\mathcal{R}(P)$ perpendicular to $\mathcal{R}(P)$.

(I think this topic is the most difficult/abstract we'll encounter this semester. If you can keep up with this, you're in good shape.)

Visual Interpretation

Oblique projections are caused when we shine the light from some other direction.

We're more interested in orthogonal projectors this semester.

Projector Facts

- If $v \in \mathcal{R}(P)$, then it lies on its own shadow, so applying the projector gives v back.

  Proof: $v = Px$ for some x because $v \in \mathcal{R}(P)$. So $Pv = P^2x = Px = v$.

- If $v \notin \mathcal{R}(P)$, then there is some part of v that lives outside this space.

  That is $v \neq Pv$.

  Where does $Pv - v$ live?

Projector Facts

To see that $Pv - v$ lives in $\mathcal{N}(P)$:

$$P(Pv - v) = P^2v - Pv = Pv - Pv = 0$$

Projector Facts

If P is a projector, $I - P$ is also a projector:

$$(I - P)^2 = I - 2P + P^2 = I - P.$$

Where does this projector project? Math 22 people....

- For any v, $(I - P)v = v - Pv$, which we said lives in $\mathcal{N}(P)$. Thus, $\mathcal{R}(I - P) \subseteq \mathcal{N}(P)$.
- For any item w in $\mathcal{N}(P)$, $Pw = 0$. But if $Pw = 0$, $(I - P)w = w$, so $w \in \mathcal{R}(I - P)$. Thus, $\mathcal{N}(P) \subseteq \mathcal{R}(I - P)$.
- Therefore, $\mathcal{R}(I - P) = \mathcal{N}(P)$.

Complementary Projectors

We say $I - P$ is a complementary projector to P because it projects onto $\mathcal{N}(P)$ whereas P projects onto $\mathcal{R}(P)$, and from linear algebra we know

$$\mathcal{R}(P) \bigcap \mathcal{N}(P) = \{0\}$$

That is, the projector separates $\mathbb{C}^m$ into two spaces.

Example

$$\text{Let} \boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- What space does P project onto?
- What space does $I - P$ project onto?

Note this is NOT an ORTHOGONAL MATRIX!

Example 2

Let

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1/\sqrt{2}} \\ \mathbf{1/\sqrt{2}} \end{bmatrix} \begin{bmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{.5} & \mathbf{.5} \\ \mathbf{.5} & \mathbf{.5} \end{bmatrix}$$

- What space does P project onto?
- What space does $I - P$ project onto?

This is not orthogonal, either!

Vector Decomposition

If S_1 and S_2 are 2 subspaces of $\mathbb{C}^m$ with $S_1 \bigcap S_2 = \{0\}$, then S_1 and S_2 are complementary subspaces and it is *always* possible to decompose an arbitrary vector $v \in \mathbb{C}^m$ uniquely as

$$v = \underbrace{v_1}_{\text{in } S_1} + \underbrace{v_2}_{\text{in } S_2} .$$

Proof: Let P be a projector onto $\mathcal{R}(P)$. Then

$$v = Pv - Pv + v = Pv + (I - P)v = v_1 + v_2.$$

Orthogonal Projectors

An orthogonal projector is one that projectors onto a subspace S_1 along a space S_2, where S_1 and S_2 are orthogonal.

Mathematically, if P is a projector (i.e. $P^2 = P$) then it's orthogonal if we also have $P^* = P$.

In summary, orth. projector is matrix P that satisfies:

- $P^2 = P$
- $P^* = P$

Example

$$\text{Let } \boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

P is an orth. projector onto $x-y$ plane (i.e. onto span$\{e_1, e_2\}$).

Note this is NOT an ORTHOGONAL MATRIX!

Example 2

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1/\sqrt{2}} \\ \mathbf{1/\sqrt{2}} \end{bmatrix} \begin{bmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{.5} & \mathbf{.5} \\ \mathbf{.5} & \mathbf{.5} \end{bmatrix}$$

P is an orthogonal projector onto the line $x = y$. But it is NOT an orthogonal matrix. In fact, note the rank of P is 1.

Theorem 6.1

A projector P is orthogonal *if and only if* $P = P^*$.

(The 250NLA students should read the proof.)

Projector $I - P$

$$I - P$$

is also an orthogonal projector. Furthermore, the space $\mathcal{R}(P)$ is orthogonal to $\mathcal{R}(I - P)$. That is, if $w \in \mathcal{R}(P)$ and $y \in \mathcal{R}(I - P)$, then $w^* y = 0$.

Proof: On board.

Projection Via Orthonormal Basis

Let $\hat{Q}$ be an $m \times n$ matrix with $m > n$ and orthonormal columns.

Then $P = \hat{Q}\hat{Q}^*$ is an orthogonal projector onto the span of the columns of $\hat{Q}$.

Proof: Trivial! On board.

Projection Via Orthonormal Basis

- Note that since $\hat{Q}$ is tall and skinny $\hat{Q}\hat{Q}^*$ is NOT the identity matrix!!!

- Note that $I - \hat{Q}\hat{Q}^*$ is also an orthogonal projector.

- However, $\hat{Q}^*\hat{Q}$ IS the identity matrix, since the columns of $\hat{Q}$ are orthonormal.

Example 2 again

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1/\sqrt{2}} \\ \mathbf{1/\sqrt{2}} \end{bmatrix} \begin{bmatrix} \mathbf{1/\sqrt{2}} & \mathbf{1/\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{.5} & \mathbf{.5} \\ \mathbf{.5} & \mathbf{.5} \end{bmatrix}$$

Clearly, P is not an orthogonal matrix, and it is a rank-1 matrix comprised of the outer product of 2 orthogonal vectors.

The matrix $\boldsymbol{I} - \boldsymbol{P} = \begin{bmatrix} \mathbf{.5} & \mathbf{-.5} \\ \mathbf{-.5} & \mathbf{.5} \end{bmatrix}$ is an orthogonal projector onto the line $y = -x$.

Projectors from 1 vector

Example 2 is a special case of an orthogonal projector onto a single direction.

- If q is an m-length vector with $\|q\|_2 = 1$, then $P = qq^*$ is a rank-1 orthogonal projector onto span$\{q\}$.
- You can show $I - P = I - qq^*$ will have rank m-1 . This makes sense, because projectors “divide” up the space $\mathbb{C}^m$.

Projectors from 1 vector

Suppose $\|a\|_2 \neq 1$. We know that the vector $\frac{1}{\|a\|_2}a$ does have unit norm.

Therefore,

$$P = (\frac{1}{\|a\|_2}a)(\frac{1}{\|a\|_2}a)^* = (\frac{1}{\|a\|_2^2})aa^* = \frac{aa^*}{a^*a}$$

is an orthogonal projector of rank 1.

Projectors from multiple vectors

Given n linearly independent vectors $a_1, \ldots, a_n$ in $\mathbb{C}^m$, $m > n$, it can be shown that with $A = [a_1, \ldots, a_n]$,

$$P = A(A^*A)^{-1}A^*$$

is the orthogonal projector onto $\mathcal{R}(A) = \text{span}\{a_1, \ldots a_n\}$.

It's the multidimensional generalization of the rank-1 formula. (250NLA students should understand why, bottom page 46)

Final Comments

- If $A = \hat{Q}$, we get back the same formula as before.
- $\hat{Q}\hat{Q}^*$ is sometimes referred to as a rank-n outer product if $\hat{Q}$ has n columns. This is because

$$\hat{Q}\hat{Q}^* = q_1 q_1^* + q_2 q_2^* + \cdots + q_n q_n^*$$

  is a sum of n, rank-1 projectors.
- Therefore, $I - \hat{Q}\hat{Q}^*$ will have rank $m - n$.

Example 1 revisited

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{e_1^T} \\ \boldsymbol{e_2^T} \end{bmatrix}.$$

P has rank 2.

$$I - P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

has rank 1.

Final Comments, Con't

- We will see that forming $A(A^*A)^{-1}A^*$ is a REALLY BAD idea. Instead, compute orthonormal bases for A first. Could use SVD, but we'll learn a different way next.
- Orthogonal projections are used in PCA applications
- Orthogonal projections arise naturally in data fitting (linear least squares)

- If λ is an eigenvalue of orthogonal projector P, then $|\lambda| = 1$ or 0.
- What is $\|P\|_2$ if P is an orth. projector?