MATH 128: HOMEWORK 1 SOLUTIONS

Problem 1. (10 pts) Let B be a 4×4 matrix to which we apply the following operations:

(1) double column 1
(2) halve row 3
(3) add row 3 to row 1
(4) interchange columns 1 and 4
(5) subtract row 2 from each of the other rows
(6) replace column 4 by column 3
(7) delete column 1 (so that the dimension is reduced by 1)

(a) write the result as a product of 8 matrices.

Solution. Let

$$A_1 := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A_3 := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_4 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} A_5 := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} A_6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_7 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then,

$$A_5 A_3 A_2 B A_1 A_4 A_6 A_7$$

□

(b) Write the result as a product of ABC (same B) of 4 matrices.

Solution.

$$A = A_5 A_3 A_2 = \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$C = A_1 A_4 A_6 A_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

□

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Problem 2. (12 pts) Suppose I have collected the following data:

t_i	$g(t_i)$
1	4.2
2	9.1
3	17.75
4	30

We would like to fit this data by a quadratic (use the form $f(t) = a_0 + a_1 t + a_2 t^2$). In other words, we would ideally want $f(t_i) = g(t_i)$ for $i = 1, 2, 3, 4$.

(a) Set up the matrix-vector equation $Ax = b$, defining the entries in A, x and b, that one would have to "solve" to determine the unknown coefficients in the quadratic.

Solution.

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} 4.2 \\ 9.1 \\ 17.75 \\ 30 \end{bmatrix}}^{b}$$

□

(b) Are the columns of the matrix A linearly independent? Why or why not?

Solution. The columns of A are linearly independent. You may row reduce A to show there are is a pivot in each column or you may show that the only solution to $Ax = 0$ is $x = 0$. □

(c) Under what condition(s) will there be an exact solution to this matrix-vector equation? Justify your answer. Use a linear algebra or geometric argument.

Solution. Linear Algebra Argument

If $b \in \text{Range}(A)$, i.e. we can write b as a linear combination of the columns of A, then a solution exists. This solution will be unique since the columns of A are linearly independent.

Row-reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 3 & 9 & b_3 \\ 1 & 4 & 16 & b_4 \end{array}\right] \sim \left[\begin{array}{ccc|l} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 2 & b_3 - b_1 - 2(b_2 - b_1) \\ 0 & 0 & 0 & b_4 - b_1 - 3(b_2 - b_1) - 3(b_3 - b_1 - 2(b_2 - b_1)) \end{array}\right]$$

We find that if

$$b_4 = b_1 - 3b_2 + 3b_3$$

then the system is consistent. For our given b

$$b_2 - 3b_2 + 3b_3 = 4.2 - 3(9.1) + 3(17.75) = 30.15 \neq 30$$

So in this case, there is no solution.

Geometric Argument

To define a quadratic, only three distinct points are needed. We are given 4 distinct points. So, unless these four points, $(t_i, g(t_i))$, already sit on a quadratic, there is no way to fit them to a quadratic. □

Problem 3. (12 pts) Let A be an $m \times m$ matrix and suppose we can write $A = QR$, where Q and R are both $m \times m$ matrices, Q is invertible, and R is upper triangular (i.e. all entries below the main diagonal are 0).

(a) Suppose R is **not** invertible. Given an *arbitrary* vector b of length m, can we grarantee there will be at least one solution to the equation $Ax = b$? Why or why not?

Solution. No. Since R is not invertible, there exists at least one $0 \neq x \in \text{Null}(R)$. Then

$$Rx = 0 \Longrightarrow Ax = QRx = Q0 = 0$$

Therefore $x \in \text{Null}(A)$. Then $\dim \text{Null}(A) \geq 0$. By the rank nullity theorem, this implies that $\dim \text{Range}(A) < m$. So A does not span $\mathbb{R}^m$ and thus there are elements $b \in \mathbb{R}^m - \text{Range}(A)$. □

(b) Based on your answer above, when R is not invertible, is $\text{Range}(A) = \mathbb{R}^m$? Explain.

Solution. No. As shown in the above proof $\dim \text{Range}(A) < m$, therefore A does not span $\mathbb{R}^m$. □

(c) Still suppose R is not invertible. Assume b is such that $Ax = b$. Is this x the ONLY solution to $Ax = b$? Explain.

Solution. No. Assume b is such that $Ax = b$. By the Rank Nullity theorem, we know dim Range A + dim Null A = m. By (b) , if R is not invertible, we know $\dim \text{Range}(A) < m$ hence $\dim \text{Null}(A) > 0$. So there is some $x_0 \neq 0$ such that $Ax_0 = 0$. Thus

$$A(x + 0) = Ax + 0 = b$$
$$A(x + x_0) = Ax + 0 = b$$

Hence both x and $x + x_0$, where $x \neq x + x_0$, are solutions to $Ax = b$. □

(d) If R is not invertible and we know it is upper triangular, what must be true about at least one of the diagonal entries and why?

Solution. At least one of the diagonal entries is 0. This is because if R is not invertible then R has fewer than m pivots by the Invertible Matrix Theorem, therefore at least one of the pivot positions is 0 which cooresponds to having a zero for at least one diagonal entry. □

Problem 4. (16 pts) Define

$$A := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Let $\theta = \frac{\pi}{4}$. Thus

$$A = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(a) Suppose that x is any point on the line with slope 1 through the origin. What is the effect (geometrically) of computing Ax?

Solution. Ax rotates the point x clockwise $\pi/4$ radians while preserving the vector length. For a point on the given line we have

$$Ax = \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ x\end{bmatrix} = \sqrt{2x^2}\begin{bmatrix}1\\ 0\end{bmatrix} = \|x\|_2\begin{bmatrix}1\\ 0\end{bmatrix}$$

□

(b) Now let x lie on the line with slope -1 through the origin. Describe Ax.

Solution. As above, Ax rotates the point x clockwise $\pi/4$ radians while preserving the vector length. For a point on the given line we have

$$Ax = \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ -x\end{bmatrix} = \sqrt{2x^2}\begin{bmatrix}0\\ 1\end{bmatrix} = \|x\|_2\begin{bmatrix}0\\ 1\end{bmatrix}$$

□

(c) Clearly the vectors

$$v_1 := \begin{bmatrix}1\\ 1\end{bmatrix}, v_2 := \begin{bmatrix}-1\\ 1\end{bmatrix}$$

are independent since they are not multiples of one another. Therefore, any vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors. Use this fact to describe the effect of Ax for any x in $\mathbb{R}^2$.

Solution. Let

$$x = c_1\begin{bmatrix}1\\ 1\end{bmatrix} + c_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

then

$$\begin{aligned}
Ax &= \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\left(c_1\begin{bmatrix}1\\ 1\end{bmatrix} + c_2\begin{bmatrix}-1\\ 1\end{bmatrix}\right)\\
&= \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\left(c_1\begin{bmatrix}1\\ 1\end{bmatrix} + c_2\begin{bmatrix}-1\\ 1\end{bmatrix}\right)\\
&= \frac{\sqrt{2}}{2}\left(c_1\begin{bmatrix}2\\ 0\end{bmatrix} + c_2\begin{bmatrix}0\\ 2\end{bmatrix}\right)\\
&= \sqrt{2}\left(c_1\begin{bmatrix}1\\ 0\end{bmatrix} + c_2\begin{bmatrix}0\\ 1\end{bmatrix}\right)\\
&= \sqrt{2}\begin{bmatrix}c_1\\ c_2\end{bmatrix}
\end{aligned}$$

Thus Ax projects x onto the new coordinate system given by v_1 and v_2. □

(d) Use only your answers from the first two parts of this question to find A^{-1}.

Solution. By (a) we see that

$$A\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\ 0\end{bmatrix}$$
$$A\begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ \sqrt{2}\end{bmatrix}$$

Then,

$$Az_1 = A\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$Az_2 = A\begin{bmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

Hence

$$Z = A^{-1} = \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

□

Problem 5. (5 pts) How many flops, in big-Oh, does it take to compute the outer product vw^T if v and w are length n vectors? Be sure to show all your work.

Solution.

$$\begin{bmatrix}v_1\\v_2\\ \vdots\\ v_n\end{bmatrix}\begin{bmatrix}w_1 & w_2 & \cdots & w_n\end{bmatrix} = \begin{bmatrix}v_1w_1 & v_1w_2 & \cdots & v_1w_n\\ v_2w_1 & v_2w_2 & \cdots & v_2w_n\\ \vdots & \ddots & & \\ v_nw_1 & v_nw_2 & \cdots & v_nw_n\end{bmatrix}$$

We have one multiplication per matrix entry and $n * n$ entries, so we have $\mathcal{O}(n^2)$ flops. □

Problem 6. (10 pts) Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$ and $C = AB$. If we compute C using outer products

$$C = \sum_{i=1}^{n} a_i \widetilde{b}_i$$

where $\widetilde{b}_i$ is the i^{th} row of B. Give the number of flops, in big-Oh, for computing the mtarix-matrix product in this way.

Solution. For $C = \sum_{i=1}^{n} a_i\widetilde{b}_i$ we have n outer products each $\mathcal{O}(n^2)$, which is $\mathcal{O}(n^3)$. Then we have to add $n-1$ matrices of size $n \times n$. And each matrix addition takes n^2 flops, one for each entry. Hence we have $n^3 + n^2(n-1)$ which is $\mathcal{O}(n^3)$ flops. □

Problem 7. (15 pts) If A and B are lower triangular matrices (i.e. all entries above the main diagonal are 0), their product must be lower triangular.

Proof. Let A and B be lower triangular matrices. Let $\widetilde{a}_i$ represent the i^{th} row of A and b_j represent the j^{th} column of B. Then recall that we may write the entries of $C = AB$ as the dot product below

$$c_{ij} = (AB)_{ij} = \widetilde{a}_i \cdot b_j$$

Notice that in $\widetilde{a}_i$ for all $k > i$ $a_{ik} = 0$ and in b_j for all $k < j$ $b_{kj} = 0$. Therefore, if $j > i$ there is no overlap in nonzero entires and $c_{ij} = 0$. Therefore, C is lower triangular. □