TOPOLOGICAL ENTROPY OF m-FOLD MAPS

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Abstract. We investigate the relation between preimage multiplicity and topological entropy for continuous maps. An argument originated by Misiurewicz and Przytycki shows that if every regular value of a C^1 map has at least m preimages then the topological entropy of the map is at least $\log m$. For every integer, there exist continuous maps of the circle with entropy zero for which every point has at least m preimages. We show that if in addition there is a positive uniform lower bound on the diameter of all pointwise preimage sets, then the entropy is at least $\log m$.

1. Introduction and Statement of Results

In [MP77], Misiurewicz and Przytycki proved the following estimate on topological entropy:

Theorem 1.1. *If $f\colon X \to X$ is a continuously differentiable self-map on a compact manifold, then*

$$h_{top}(f) \geq \log |\deg f|. \tag{1}$$

The differentiability hypothesis is needed, as shown by Shub's example [Shu74]: the map f of the Riemann sphere to itself which fixes 0 and ∞ and for $0 < |z| < \infty$ (and some $m \geq 2$)

$$f(z) = \frac{z^m}{2|z|^{m-1}}$$

has degree m, but for $0 < |z| < \infty$, $|f(z)| = \frac{1}{2}|z|$, so the nonwandering set of f consists of the two fixedpoints, and hence $h_{top}(f) = 0$. The map cannot be C^1 at the fixed repellor ∞, since it is not injective on any neighborhood, so differentiability would force ∞ to be a critical point, and hence have at least one attracting direction.

The estimate (1) gives no information when X is simply connected, but a closer examination of the proof in [MP77] reveals that they have actually shown a stronger result. In fact, the degree of the map enters the proof in [MP77] only as a lower bound for the cardinality of $f^{-1}[y]$ for any regular (*i.e.*, noncritical) value of f. We call a continuous map $f\colon X \to X$ (X a compact metric space) ***m*-fold** on a subset $Y \subset X$ if for every $y \in Y$ the preimage set

$$f^{-1}[y] := \{x \in X \mid f(x) = y\}$$

contains at least m distinct points; when $Y = X$, f is **globally *m*-fold**. In §2 we present the argument from [MP77] to prove

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Theorem 1.2. *If X is a compact manifold and $f\colon X \to X$ is a continuously differentiable map which is m-fold on the set of regular values, then*

$$h_{top}(f) \geq \log m.$$

In [Bob02, Bob04], the first author showed that for maps of the interval, the smoothness hypothesis is not needed; this had been conjectured by Ethan Coven for $m = 2$. We call a subset $Y \subset X$ **cocountable** if its complement $X \setminus Y$ is (at most) countable, and say that $f\colon X \to X$ is **cocountably m-fold** if it is globally 2-fold and m-fold on some cocountable subset $Y \subseteq X$.

Theorem 1.3 ([Bob04])**.** *The topological entropy of any cocountably m-fold map $f\colon [0,1] \to [0,1]$ satisfies*

$$h_{top}(f) \geq \log m.$$

This result is surprizingly delicate, as there is a simple example (shown to us by Ethan Coven, who attributed it to Peter Raith) of a map $f\colon [0,1] \to [0,1]$ exhibiting phenomena similar to Shub's example: it is m-fold (for an arbitrarily chosen $m \in \mathbb{N}$) except at $y = 1$, which has a single preimage point, but its nonwandering set consists of the fixed endpoints, so that the entropy is zero. To construct such an example, let $\{y_i\}_{i=0}^{\infty}$ be a strictly increasing sequence in $[0,1]$ with $y_0 = 0$ and $\lim y_i = 1$, and set $I_j = [y_j, y_{j+1}]$. For $j > 0$, define f to map I_j onto I_{j-1} in an m-fold way, taking y_j (*resp.* y_{j+1}) to y_{j-1}(*resp.* y_j). Define f to map I_0 into itself so that $f(x) < x$ for every interior point of I_0 and $f(y_0) = f(y_1) = 0$. Finally, set $f(1) = 1$. Then clearly $f(x) < x$ for $0 < x < 1$, so $f^i(x) \to 0$ for $x < 1$, and every $y < 1$ has m preimages.

We note that the inequality

$$h_{top}(f) \geq \log m \tag{2}$$

is trivial to prove for any globally m-fold map (on any metric space) which has *uniform separation of preimages*—that is, for each $y \in X$, $f^{-1}[y]$ is ε-separated for some uniform $\varepsilon > 0$ (*i.e.*, $\operatorname{dist}(x, x') \geq \varepsilon$ if $x \neq x' \in f^{-1}[y]$).

The bulk of the present paper is involved in an examination of the entropy of continuous m-fold maps of the circle $S := \mathbb{R}/\mathbb{Z}$ to itself. By identifying the endpoints of $[0,1]$ in the example sketched above, one obtains a globally m-fold map of the circle with entropy zero, so to obtain nontrivial entropy estimates we need an additional condition. We will say that f has **no small preimage sets** if the diameter of all preimage sets $f^{-1}[y]$ is bounded below by a positive constant $\alpha > 0$. Note that by contrast with uniform separation of preimages, we require only that for each $y \in S$ there exist two points x, x' with $f(x) = f(x') = y$ and $\operatorname{dist}(x, x') \geq \alpha$; the other $m-2$ preimages of y can be arbitrarily close to each other or to x or x'. Of course, this automatically implies that f is globally 2-fold, and that (2) holds with $m = 2$. Our result is

Theorem 1.4. *If $f\colon S \to S$ is continuous, cocountably m-fold, and has no small preimage sets, then*

$$h_{top}(f) \geq \log m.$$

A special case of our result is that if f has degree at least 2 and is globally m-fold, then (2) holds: a map of degree d on the circle has $|d|$ uniformly separated preimages for every $y \in S$, and (2) is immediate for $m = |d|$. Our result gives further information only if $m > |d|$.

A standard method for establishing estimates like (2) (often in the context of "horseshoes"$-$*cf* [ALM00, Proposition 4.3.2]) is to find a collection $\mathcal{H} = \{H_1, \ldots, H_m\}$ of disjoint closed sets, each f-covering their union:

$$f(H_i) \supseteq \mathbb{H} := H_1 \cup \cdots \cup H_m \quad \text{for } i = 1, \ldots, m. \tag{3}$$

Then the restriction of f to the nonempty closed invariant set

$$\mathbb{D} := \bigcap_{k=0}^{\infty} f^{-k}[\mathbb{H}] \tag{4}$$

has the full (one-sided) m-shift $\sigma\colon \Omega_m \to \Omega_m$ as a factor, via the coding map $\mathbf{a}\colon \mathbb{D} \to \Omega_m$ which assigns to $x \in \mathbb{D}$ its **itinerary**

$$\mathbf{a}(x) = a_0 a_1 \cdots \in \Omega_m := \{1, \ldots, m\}^{\mathbb{N}},$$

where the i^{th} **address** a_i is defined by

$$f^i(x) \in H_{a_i}. \tag{5}$$

Then (2) follows by standard arguments: Given $0 < \varepsilon < \min_{i \neq j} \operatorname{dist}(H_i, H_j)$, we can, for each word

$$w = a_0 ... a_{n-1} \in \Omega_m(n) := \{1, \ldots, m\}^n,$$

pick an element $x_w \in \mathbb{D}$ whose itinerary begins with w. This collection of m^n points is (n, ε)-separated, and (2) follows.

Our approach, following [Bob02, Bob04], is to attempt such a construction using *sections* of f. If $f\colon S \to S$ is m-fold on $Y \subseteq S$, we can choose for each $y \in Y$ a collection of m preimages $\psi_i(y) \in f^{-1}[y]$, $i = 1, \ldots, m$. The resulting map

$$\psi\colon Y \to \mathop{\times}_{i=1}^{m} S := S \times \cdots \times S$$

is an ***m*-section** for f on Y. Given an m-section ψ on $Y \subseteq S$, we can define closed subsets H_i, $i = 1, \ldots, m$ by

$$H_i = H_i(\psi) := \operatorname{clos} \{\psi_i(y) \mid y \in Y\}. \tag{6}$$

When Y is dense in S, we can guarantee the f-covering condition (3), but we cannot guarantee that the sets H_i are disjoint.

In §3 we formulate a weakening of the notion discussed above, which we call an *m-shift system*. In §4 we formulate conditions on such a system that guarantee the estimate (2) (Theorem 4.8) and in §§5, 6 we show how to construct an m-section for any cocountably m-fold map of the circle with no small preimage sets whose related m-shift system (defined by (6)) satisfies these conditions.

Finally, we note that an estimate complementary to theorems 1.3 and 1.4 is implicit in [ALM00]:

Remark 1.5. *Let $X = [0,1]$ or S, and suppose $f\colon X \to X$ is continuous such that the preimage set $f^{-1}[y]$ has at most $M < \infty$ components for each $y \in X$. Then*

$$h_{top}(f) \leq \log M.$$

To see this, note that our assumption implies for each $k \in \mathbb{N}$ that $f^{-k}[y]$ has at most M^k components for each $y \in X$. According to [ALM00, Theorem 4.3.5], if $h_{top}(f) > 0$, there exist $k_n \to \infty$ and $s_n \in \mathbb{N}$ with $\lim \frac{1}{k_n} \log s_n = h_{top}(f)$ such that f^{k_n} has an *s_n-horseshoe*— a collection of closed intervals H_i, $i = 1, \ldots, s_n$, intersecting (if at all) only at their endpoints and satisfying (3) with f replaced by

f^{k_n}. Given a horseshoe, a point $y \in \mathbb{D}$ which is not the image of any endpoint has at least one preimage component in each H_i. Thus, there exist points y for which $f^{-k_n}[y]$ has at least s_n components, and so $s_n \leq M^{k_n}$. It follows that

$$h_{top}(f) = \lim \frac{1}{k_n} \log s_n \leq \log M.$$

Remark 1.5 as well as our argument is a slight variation on [ALM00, Theorem 4.3.14].

2. Smooth Maps

In this section we use the argument from [MP77] to prove

Theorem 1.2 *If X is a compact manifold, then for any continuously differentiable map $f \colon X \to X$ which is m-fold at all regular values,*

$$h_{top}(f) \geq \log m.$$

A Riemannian metric on X defines a volume function on sufficiently small balls in X, and hence a finite "Lebesgue"-like Borel measure μ on X which is conformal with respect to the Jacobian $Jf(x)$ (essentially the absolute value of the determinant of partial derivatives, in local coordinates): every regular (*i.e.*, not critical) point $x \in X$ has a neighborhood $U(x)$ on which f is injective, and

$$\mu(f[U(x)]) = \int_{U(x)} Jf d\mu.$$

Given $\varepsilon > 0$, let

$$B_\varepsilon := \{x \in X \mid Jf(x) \geq \varepsilon\}$$

which is a closed and (for $\varepsilon > 0$ sufficiently small) nonempty subset of X. Define a set-valued map P_ε on X as follows: if $f^{-1}[y]$ contains at least m distinct elements of B_ε, choose $P_\varepsilon(y)$ to be such a collection; if not, there exists at least one point in $f^{-1}[y] \setminus B_\varepsilon$ and choose $P_\varepsilon(y)$ to consist of one such point. Using this, inductively define, for each $y \in X$, a sequence of subsets $Q^\varepsilon_j(y) \subset f^{-j}[y]$, $j = 0, 1, ...$, by

$$\begin{aligned} Q^\varepsilon_0(y) &:= \{y\} \\ Q^\varepsilon_{j+1}(y) &:= \bigcup \{P_\varepsilon(y') \mid y' \in Q^\varepsilon_j(y)\}. \end{aligned}$$

We wish to establish two properties of these sets. First, let $\delta > 0$ (depending on ε) be a Lebesgue number for the cover of the compact set B_ε by the open sets $U(x)$, so that

$$x', x'' \in B_\varepsilon,\ 0 < \operatorname{dist}(x', x'') < \delta \Rightarrow f(x') \neq f(x''). \tag{7}$$

Lemma 2.1. *For each $y \in X$, $n \in \mathbb{N}$ and δ as in (7), $Q^\varepsilon_n(y)$ is (n, δ)-separated.*

Proof. Note that if x and x' are distinct points of $Q^\varepsilon_j(y)$ with $f(x) = f(x') = y'$, then $y' \in Q^\varepsilon_{j-1}(y)$ and $x, x' \in P_\varepsilon(y') \subset B_\varepsilon$, so by (7), $\operatorname{dist}(x, x') \geq \delta$. But for any pair z, z' of distinct points in $Q^\varepsilon_n(y)$, there is a unique $j < n$ such that $f^j(z) \neq f^j(z')$ but $f^{j+1}(z) = f^{j+1}(z')$; applying the preceding observation we have $\operatorname{dist}(f^j(z), f^j(z')) \geq \delta$, as required. □

For the second property, given any point $x \in X$ and any subset $Z \subset X$, for each $n \in \mathbb{N}$ we denote the ***n*-sojourn time** of x in Z by

$$\theta(x, n, Z) := \operatorname{card}\{i < n \mid f^i(x) \in Z\}. \tag{8}$$

Now, given $x \in X$ and $n \in \mathbb{N}$, denote the n-sojourn time of x in B_ε by

$$\theta_n(x) := \theta(x, n, B_\varepsilon).$$

Lemma 2.2. *If $\theta_n(z) \geq k$ for all $z \in Q_n^\varepsilon(y)$, then*

$$\operatorname{card} Q_n^\varepsilon(y) \geq m^k.$$

Proof. Since there is nothing to prove for $k = 0$, assume $k \geq 1$.

Let

$$R := \bigcup_{j=1}^{n} Q_j^\varepsilon(y) \times \{j\}$$

and for $\ell = 1, \ldots, n$

$$R(\ell) := \{(x, j) \in R \mid \theta_j(x) = \ell \text{ and } \theta_{j-1}(f(x)) < \ell\}$$

while for $\ell = 0$

$$R(0) := \{(y, 0)\}.$$

Note that

$$\bigcup_{\ell=1}^{n} R(\ell) = \bigcup_{j=1}^{n} [Q_j^\varepsilon(y) \cap B_\varepsilon] \times \{j\}.$$

If $(x, j) \in R$ with $\theta_j(x) > 0$, let

$$i(x, j) = \min\{i \geq 0 \mid f^i(x) \in B_\varepsilon\}$$

and set

$$i(x, j) = j \text{ if } \theta_j(x) = 0.$$

Then $p : (x, j) \mapsto (f^{i(x,j)}(x), j - i(x, j))$ maps R to $\bigcup_{\ell=0}^{n} R(\ell)$ and is injective on each stratum $Q_j^\varepsilon(y) \times \{j\}$.

Now, for $\ell = 1, \ldots, n$ define $q \colon R(\ell) \to R(\ell - 1)$ by

$$q(x, j) = p(f(x), j - 1).$$

Since $(x, j) \in R(\ell)$, $\ell \geq 1$ implies $x \in B_\varepsilon$, and hence $P_\varepsilon(f(x)) = Q_j^\varepsilon(y) \cap f^{-1}[f(x)]$ has cardinality m, we see that q is m-to-one. Furthermore, for $\ell \leq k$, it maps $R(\ell)$ onto $R(\ell - 1)$; it follows that $\operatorname{card} R(k) = m^k$. But for $z \in Q_n^\varepsilon(y)$, there is a unique point $x = f^i(z)$, $0 \leq i < n$ with $(x, n - i) \in R(k)$, so the lemma follows. $\square$

Combining these lemmas, we get

Proof of Theorem 1.2:

Pick $0 < \alpha < 1$ and let $L := \max\{Jf(x) \mid x \in X\}$; fix $\varepsilon > 0$ sufficiently small that

$$L^\alpha \varepsilon^{1-\alpha} < 1. \tag{9}$$

For $n \in \mathbb{N}$ define

$$A_{\alpha,\varepsilon}(n) := \{x \in X \mid \theta(x, n, B_\varepsilon) \leq \alpha n\}.$$

If $x \in A_{\alpha,\varepsilon}(n)$, then by (9)

$$Jf^n(x) = \prod_{i=0}^{n-1} Jf(f^i(x)) < L^{\alpha n} \varepsilon^{(1-\alpha)n} < 1,$$

so that, if $A_{\alpha,\varepsilon}(n)$ is not μ-null, then $\mu(f^n(A_{\alpha,\varepsilon}(n))) < \mu(A_{\alpha,\varepsilon}(n))$; in any case, then,

$$C_n := X \setminus f^n(A_{\alpha,\varepsilon}(n))$$

has positive μ-measure, and hence is nonempty. For any $y \in C_n$, we have $\theta_n(x) > \alpha n$ for all $x \in Q_n^{\varepsilon}(y)$, so by Lemma 2.2 and Lemma 2.1 this represents an (n, δ)-separated set of cardinality at least $m^{\alpha n}$. It follows that

$$h_{top}(f) \geq \alpha \log m$$

but since α can be picked arbitrarily close to 1, the theorem follows. □

3. Shift Systems

For the remainder of this paper, we shall concentrate on maps $f \colon S \to S$, where S is the circle. We refer to a collection

$$\mathcal{H} = \{H_1, ..., H_m\}$$

of m nonempty (but not necessarily closed or disjoint) sets satisfying (3) as an ***m*-shift system** for f[1]. The **address set** of $x \in S$ is

$$\alpha(x) := \{a \in \{1, \dots, m\} \,|\, x \in H_a\}$$

and its cardinality $\eta(x)$ is the **multiplicity** of $\mathcal{H}$ at x. The set of points with positive multiplicity is $\mathbb{H}$, defined in (3); we define the **kernel** (*resp.* **center**) of $\mathcal{H}$ to be the set of points with multiplicity greater than one (*resp.* equal to m)

$$\begin{aligned} \mathfrak{K}(\mathcal{H}) := \{x \,|\, \eta(x) > 1\} &= \bigcup_{i \neq j} H_i \cap H_j \\ \mathfrak{Z}(\mathcal{H}) := \{x \,|\, \eta(x) = m\} &= \bigcap_{i=1}^{m} H_i. \end{aligned}$$

We also define the **core** of $\mathcal{H}$ as the set of points whose orbit remains in $\mathfrak{Z}(\mathcal{H})$ for all time:

$$\mathfrak{Z}_0(\mathcal{H}) := \{x \in S \,|\, f^k(x) \in \mathfrak{Z}(\mathcal{H}) \text{ for all } k\} = \bigcap_{k=0}^{\infty} f^{-k}[\mathfrak{Z}(\mathcal{H})].$$

Obviously, $\mathfrak{Z}_0(\mathcal{H}) \subseteq \mathfrak{Z}(\mathcal{H}) \subseteq \mathfrak{K}(\mathcal{H}) \subseteq \mathbb{H}$. We will call the shift system $\mathcal{H}$ **nontrivial** if $H_i \setminus \mathfrak{Z}_0(\mathcal{H}) \neq \emptyset$ for all i, and **closed** if each H_i is a closed subset of S.

For $n \in \mathbb{N}$, the ***n*-itinerary set** of $x \in S$ is the subset of $\Omega_m(n)$ defined via (5):

$$\Omega(x)(n) := \{a_0 \dots a_{n-1} \in \Omega_m(n) \,|\, f^i(x) \in H_{a_i} \text{ for } i < n\} = \mathsf{X}_{i=0}^{n-1} \alpha(f^i(x)).$$

The set of points for which $\Omega(x)(n) \neq \emptyset$

$$\mathbb{D}_n := \bigcap_{i=0}^{n-1} f^{-i}[\mathbb{H}]$$

is the union of the sets

$$\Pi(w) := \{x \in S \,|\, w \in \Omega(x)(n)\} = \bigcap_{i=0}^{n-1} f^{-i}[H_{w_i}] \tag{10}$$

[1][BT03], which came to our attention during the writing of this paper, treats similar ideas, in particular what we call a "closed shift system with empty center" (or more generally, with empty core).

as $w = w_0 ... w_{n-1}$ ranges over the finite collection $\Omega_m(n)$ of n-words.

Remark 3.1. *Let $\mathcal{H}$ be any m-shift system.*

1. *For each finite word $w \in \Omega_m(n)$, $\Pi(w) \neq \emptyset$.*
2. *In particular, if $\mathcal{H}$ is nontrivial, $\Pi(w)$ is not contained in $\mathfrak{Z}_0(\mathcal{H})$.*
3. *if $\mathcal{H}$ is closed, then $\Pi(w)$ is closed.*

Statements 1 and 3 are trivial consequences of (10). To see Statement 2, note that $f(\mathfrak{Z}_0(\mathcal{H})) \subset \mathfrak{Z}_0(\mathcal{H})$, so if $\mathcal{H}$ is nontrivial, then the system $\mathcal{H}'$ defined by

$$H'_i := H_i \setminus \mathfrak{Z}_0(\mathcal{H})$$

is also an m-shift system, and apply (1).

We also note the following easy fact:

Remark 3.2. *Suppose $A \subset \Omega_m$ and $x \in S$ such that*

$$\Omega(x)(k) \cap A(k) = \emptyset.$$

Then $\Omega(x)(k') \cap A(k') = \emptyset$ for all $k' \geq k$.

For $n_1 > n_2$, we have the projection maps $\Omega_m(n_1) \to \Omega_m(n_2)$ assigning to $w = w_0 ... w_{n_1-1} \in \Omega_m(n_1)$ its initial n_2-subword $w(n_2) := w_0 ... w_{n_2-1}$. The inverse limit of this system of projections is the full shift space $\Omega_m(\infty) = \Omega_m$, and the (compact) product topology on Ω_m is the inverse limit of the discrete topologies on $\Omega_m(n)$, $n \in \mathbb{N}$; it is convenient to extend our notation to the projections $\Omega_m(\infty) \to \Omega_m(n)$, $n \in \mathbb{N}$; the preimage of a word $w = w_0 ... w_{n-1} \in \Omega_m(n)$ under the appropriate projection is the **cylinder set**

$$\mathcal{C}[w] := \{\mathbf{a} = a_0 a_1 ... \in \Omega_m \mid \mathbf{a}(n) = w\}$$

of sequences in Ω_m which begin with w. This is an open and closed subset of Ω_m, and for any $\mathbf{a} \in \Omega_m$,

$$\bigcap_{n=1}^{\infty} \mathcal{C}[\mathbf{a}(n)] = \{\mathbf{a}\}. \tag{11}$$

When our shift system is closed, then we can define the **itinerary set** of $x \in S$ as

$$\Omega(x) = \Omega(x)(\infty) := \{\mathbf{a} = a_0 a_1 \cdots \in \Omega_m \mid f^i(x) \in H_{a_i} \text{ for all } i\}.$$

Remark 3.1 together with the finite intersection property insures that the set of points with $\Omega(x)$ nonempty

$$\mathbb{D} := \bigcap_n \mathbb{D}_n$$

(which agrees with (4)) and

$$\Pi(\mathbf{a}) := \{x \in S \mid \mathbf{a} \in \Omega(x)\} = \bigcap_{n=1}^{\infty} \Pi(\mathbf{a}(n))$$

are nonempty closed subsets of S; furthermore, for any $x \in S$,

$$\sigma[\Omega(x)] \subseteq \Omega(f(x))$$

with equality if $\Omega(x) \neq \emptyset$ (*i.e.*, $x \in \mathbb{D}$).

We can extend our notation in a natural way to subsets. For $Y \subset S$,

$$\Omega(Y)(n) := \bigcup_{y \in Y} \Omega(y)(n)$$

while for $A(n) \subset \Omega_m(n)$,

$$\mathcal{C}[A(n)] := \bigcup_{\mathbf{a}\in A(n)} \mathcal{C}[\mathbf{a}]$$
$$\Pi(A(n)) := \bigcup_{\mathbf{a}\in A(n)} \Pi(\mathbf{a}).$$

For $A \subset \Omega_m$, we have the natural projection to $\Omega_m(n)$, $n \in \mathbb{N}$

$$A(n) := \{\mathbf{a}(n) \,|\, \mathbf{a} \in A\} \subseteq \Omega_m(n)$$

and it is straightforward to check that if A is closed in Ω_m then

$$A = \bigcap_{n=1}^{\infty} \mathcal{C}[A(n)]$$

and

$$\Pi(A) = \bigcap_{n=1}^{\infty} \Pi(A(n)).$$

The continuity of the coding map when $\mathfrak{K}(\mathcal{H}) = \emptyset$ has a semicontinuity analogue when the shift system is closed. Recall that for any sequence A_i, $i = 1, 2, ...$

$$\limsup A_i := \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$

Lemma 3.3. *Suppose $\mathcal{H}$ is a closed m-shift system.*

(1) *For each $x \in S$, $\Omega(x)$ is a closed subset of Ω_m.*
(2) *For each nonempty (closed) set $A \subset \Omega_m(n)$, $n \in \mathbb{N}$ ($n = \infty$), $\Pi(A)$ is a nonempty closed subset of S.*
(3) *For $n \in \mathbb{N}$, if $x_i \in \mathbb{D}_n$ for $i = 1, 2, ...,$ then $\limsup \Omega(x_i)(n) \neq \emptyset$.*
(4) *The set-valued maps $x \mapsto \Omega(x)(n)$, $n \in \mathbb{N}\cup\{\infty\}$, are upper semicontinuous: if $x_i \to x$ in S, then $\limsup \Omega(x_i)(n) \subset \Omega(x)(n)$.*

Proof. (1) $\Omega(x)$ is the intersection of the closed sets $\mathcal{C}[\Omega(x)(n)]$, $n = 1, 2, ...$

(2) For $n < \infty$, $A(n)$ is finite, so $\Pi(A(n))$ is a finite union of nonempty closed sets of the form $\Pi(\mathbf{a}(n)) = \bigcap_{i=0}^{n-1} f^{-i}[H_{a_i}]$. For $n = \infty$ and A closed in Ω_m, the intersection $\Pi(A) = \bigcap_{n=1}^{\infty} \Pi(A(n))$ is closed and nonempty by the finite intersection property.
(3) The finiteness of $\Omega_m(n)$ insures that the sequence of sets $\Omega(x_i)(n)$ runs through a finite number of subsets $A_j(n) \subset \Omega_m(n)$, $j = 1, \dots, \ell$, which are nonempty if $x_i \in \mathbb{D}_n$. A nonempty collection of these occurs infinitely often, and $\limsup \Omega(x_i)(n)$ is their union.
(4) In the preceding, if $x_i \to x$ then each $\Pi(A_j(n))$ is a closed subset of S, and if it contains infinitely many x_i then it also contains x; thus for $n < \infty$, $\limsup \Omega(x_i)(n) \subset \Omega(x)(n)$.

For $n = \infty$, suppose $\mathbf{a} \in \limsup \Omega(x_i)(n)$; by an argument like the preceding, $\mathbf{a}(n) \in \Omega(x)(n)$ for all $n \in \mathbb{N}$, so

$$\{\mathbf{a}\} = \bigcap_n \mathcal{C}[\mathbf{a}(n)] \subset \bigcap_n \mathcal{C}[\Omega(x)(n)] = \Omega(x)$$

since $\Omega(x)$ is closed in Ω_m.

□

4. Entropy via Shift Systems

In this section, we formulate two new conditions on an m-shift system and show that (2) holds whenever there exists a nontrivial closed m-shift system satisfying these conditions. In §5 and 6 we shall show how such an m-shift system can be constructed for any cocountably m-fold map of the circle with no small preimage sets.

4.1. **Local Division.** Since the projection $\pi\colon \mathbb{R} \to S$ is a homeomorphism on any interval of length < 1, interval notation makes sense to denote arcs in S with distinct endpoints: for $x \neq x' \in S$, $[x, x']$ (*resp.* $[x', x]$) is the counterclockwise (*resp.* clockwise) closed arc from x to x'. By abuse of terminology, we shall refer to such arcs as "intervals". If $\operatorname{dist}(x, x') < \frac{1}{2}$, the inequality $x < x'$ is naturally interpreted to mean that $[x, x']$ is shorter than $[x', x]$.

A **full neighborhood** of $x \in S$ is any set containing $J(x, \delta) := (x - \delta, x + \delta)$ for some $\delta > 0$. We can define the notion of one-sided neighborhoods:

$$\begin{aligned} J_-(x, \delta) &:= (x - \delta, x] \\ J_+(x, \delta) &:= [x, x + \delta) \end{aligned}$$

referring to any set containing $J_-(x, \delta)$ (*resp.* $J_+(x, \delta)$) for some $\delta > 0$ as a **left** (*resp.* **right**) **neighborhood** of x.

Definition 4.1. *Suppose $P = \{p_0, \ldots, p_{t-1}\}$ ($p_i = f^i(p_0)$) is a periodic orbit for $f\colon S \to S$. We say that P **locally divides** the m-shift system $\mathcal{H}$ if we can find (full) neighborhoods $J(i) = J(p_i, \delta)$ about p_i such that*

1. *Each component* $\operatorname{int} J_\pm(i) = J_\pm(i) \setminus \{p_i\}$ *of the complement of p_i in $J(i)$ is disjoint from at least one piece H_j of $\mathcal{H}$.* [2]
2. *For each i, $f(\operatorname{int} J_-(i))$ and $f(\operatorname{int} J_+(i))$ intersect, if at all, only at p_{i+1}.*

Lemma 4.2. *If $P = \{p_0, \ldots, p_{t-1}\}$ is a periodic orbit which locally divides $\mathcal{H}$, then there exists a closed shift-invariant set $\Lambda = \Lambda(P) \subset \Omega_m$ with $\operatorname{ent}(\Lambda) := h_{top}(\sigma|\Lambda) \leq \log(m-1)$ and a neighborhood V of P such that any n-orbit segment $\{f^i(x)\}_{i=0}^{n-1}$ contained in V either terminates in P or has $\Omega(x)(n) \subset \Lambda(n)$.*

Proof. We extend the notation of Definition 4.1 so that $J(i)$ is defined for all $i \in \mathbb{N}$, with $J(i) = J(i')$ whenever $i = i' \pmod t$.

Let $V(i) \subseteq f^{-1}[J(i+1)] \cap J(i)$ be a subarc of $J(i)$, with p_i in its interior, and $V := \bigcup_{i=0}^{t-1} V(i)$. By condition 2, we can define two sequences of signs $\{s_i^-\}_{i=0}^\infty$ and $\{s_i^+\}_{i=0}^\infty$ so that $s_0^- = -$, $s_0^+ = +$ and if $x \in V_{s_i^-}$ (*resp.* $x \in V_{s_i^+}$) then $f(x) \in V_{s_{i+1}^-} \cup \{p_{i+1}\}$ (*resp.* $f(x) \in V_{s_{i+1}^+} \cup \{p_{i+1}\}$). It is possible that $s_{i+t}^\pm \neq s_i^\pm$, but $s_{i+2t}^\pm = s_i^\pm$ for all i.

For $i \in \mathbb{N}$, let

$$A_i^\pm := \alpha(J_{s_i^\pm}(i)),$$

noting that by condition 1, $\operatorname{card} A_i^\pm \leq m - 1$ for all i. For $i = 0, \ldots, t-1$,

$$\Lambda_i^\pm := \mathop{\times}_{j=0}^{\infty} A_{i+j}^\pm$$

[2] Note that p_i itself is allowed to belong to the center.

and

$$\Lambda := \bigcup_{i=0}^{t-1} [\Lambda_i^- \cup \Lambda_i^+].$$

Then $\operatorname{card} \Lambda(n) \leq 2t(m-1)^n$, so $\operatorname{ent}(\Lambda) \leq \log(m-1)$.

Furthermore, if $f^j(x) \in V(i+j) \setminus \{p_{i+j}\}$ for $j = 0, \ldots, n-1$, then clearly

$$\Omega(x)(n) \subset \Lambda_i^-(n) \cup \Lambda_i^+(n) \subset \Lambda(n)$$

as required. □

4.2. **Virtual Entropy.** For any closed f-invariant set $M \subset \mathbb{D}$, we have two "entropy" invariants: the topological entropy

$$\operatorname{ent}(M) := h_{top}(f|M)$$

and the **virtual entropy** (with respect to $\mathcal{H}$)

$$\operatorname{ent}(\Omega(M)) := h_{top}(\sigma|\Omega(M)).$$

The latter is given by

$$\operatorname{ent}(\Omega(M)) = GR\{\operatorname{card} \Omega(M)(n)\}$$

where the exponential growth rate of any sequence $\{c_n\}$ of positive numbers is

$$GR\{c_n\} := \limsup \frac{1}{n} \log c_n.$$

The relation between $\operatorname{ent}(M)$ and $\operatorname{ent}(\Omega(M))$ is delicate. For example, if the center $\mathfrak{Z}(\mathcal{H})$ (and hence the core $\mathfrak{Z}_0(\mathcal{H})$) contains a periodic orbit P, then $\operatorname{ent}(P) = 0$ but $\operatorname{ent}(\Omega(P)) = \operatorname{ent}(\Omega_m) = \log m$. We examine the situation of minimal sets outside $\mathfrak{Z}_0(\mathcal{H})$.

Remark 4.3. *For any $x \in S$ and $n \in \mathbb{N}$,*

$$\operatorname{card} \Omega(x)(n) \leq m^{\zeta_n(x)} (m-1)^{n-\zeta_n(x)},$$

where

$$\zeta_n(x) := \theta(x, n, \mathfrak{Z}(\mathcal{H}))$$

is the n-sojourn time of x in $\mathfrak{Z}(\mathcal{H})$ (as defined in (8)).

(Of course, for $x \in S \setminus \mathbb{D}_n$, $\Omega(x)(n) = \emptyset$ and the Remark is vacuous.)

The following lemma is not, strictly speaking, needed for our situation, since we will prove separately that we can get the center $\mathfrak{Z}(\mathcal{H})$ finite. However, to put the relatively abstract treatment of this section in line with [Bob04], where the center can be infinite, we show that it suffices to know that the core $\mathfrak{Z}_0(\mathcal{H})$ is finite, and locally divides $\mathcal{H}$. Recall that by definition any f-invariant subset of $\mathfrak{Z}(\mathcal{H})$ is actually contained in $\mathfrak{Z}_0(\mathcal{H})$.

Lemma 4.4. *Suppose $\mathfrak{Z}_0(\mathcal{H})$ is finite and every periodic orbit in $\mathfrak{Z}_0(\mathcal{H})$ locally divides $\mathcal{H}$. Then there exists $\zeta \in \mathbb{N}$ such that any orbit segment of length at least ζ which is contained in $\mathfrak{Z}(\mathcal{H})$ terminates in a periodic point of $\mathfrak{Z}_0(\mathcal{H})$.*

Proof. Note that local division implies each periodic point in $\mathfrak{Z}_0(\mathcal{H})$ is isolated in $\mathfrak{Z}(\mathcal{H})$, so by finiteness of $\mathfrak{Z}_0(\mathcal{H})$ the complement in $\mathfrak{Z}(\mathcal{H})$ of the periodic points in $\mathfrak{Z}_0(\mathcal{H})$ is a closed subset of $\mathfrak{Z}(\mathcal{H})$.

Assume ζ does not exist, and let $\{x_k\}_{k=0}^{\infty}$ be a sequence of points such that $f^i(x_k) \in \mathfrak{Z}(\mathcal{H})$ are distinct points for $0 \leq i \leq k$ (for each fixed k); passing to a subsequence, we can assume that $x_k \to x \in \mathfrak{Z}(\mathcal{H})$. By continuity, $f^i(x) \in \mathfrak{Z}(\mathcal{H})$ for all i—that is, $x \in \mathfrak{Z}_0(\mathcal{H})$. Pick n so that $f^n(x)$ is periodic; then $f^n(x) = \lim f^n(x_k)$. Since $f^n(x)$ is isolated in $\mathfrak{Z}_0(\mathcal{H})$, this means that $f^n(x_k)$ eventually all equal $f^n(x)$, and hence are all periodic with some fixed period, contradicting the assumption that the orbit segment of $f^n(x_{n+k})$ consists of k distinct points. □

Proposition 4.5. *Suppose the conclusion of Lemma 4.4 holds. Then there exists $\beta < \log m$ such that any periodic orbit $P \subset \mathbb{D}$ not contained in $\mathfrak{Z}(\mathcal{H})$ (i.e., disjoint from $\mathfrak{Z}_0(\mathcal{H})$) has virtual entropy*

$$\operatorname{ent}(\Omega(P)) \leq \beta.$$

Proof. For $t \geq 1$, set

$$\beta(m,t) := \frac{1}{t}\log(m-1) + (1-\frac{1}{t})\log m \tag{12}$$

and

$$\beta := \beta(m, 2\zeta) \tag{13}$$

where ζ is as in Lemma 4.4. Note that $\beta(m,t)$ is between $\log(m-1)$ and $\log m$, and converges monotonically to $\log m$ as $t \to \infty$. In particular,

$$\log(m-1) < \beta < \log m. \tag{14}$$

Let t be the least period of P. Since $P \not\subset \mathfrak{Z}(\mathcal{H})$, we have $\zeta_{nt}(p) \leq (t-1)n$ for each $p \in P$ and $n \in \mathbb{N}$. Thus by Remark 4.3

$$\operatorname{card}\Omega(P)(nt) \leq t \cdot [m^{(t-1)n}(m-1)^n]$$

and

$$\operatorname{ent}(\Omega(P)) = \lim_{n\to\infty} \frac{1}{nt}\log[\operatorname{card}\Omega(P)(nt)] \leq \beta(m,t).$$

For $t \leq \zeta$, $\beta(m,t) < \beta$ and we are done.

For $t > \zeta$, suppose $k\zeta < t \leq (k+1)\zeta$ for some $k \in \mathbb{N}$. Since p cannot spend ζ consecutive times in $\mathfrak{Z}(\mathcal{H})$, we have $\zeta_{nt}(p) \leq n(t-k)$, and so

$$\operatorname{card}\Omega(P)(nt) \leq t \cdot [m^{n(t-k)}(m-1)^{nk}]$$

so

$$\operatorname{ent}(\Omega(P)) \leq \beta(m, \frac{t}{k}).$$

But since $t \leq (k+1)\zeta$,

$$\frac{t}{k} \leq (1+\frac{1}{k})\zeta < 2\zeta$$

implying

$$\beta(m, \frac{t}{k}) \leq \beta(m, 2\zeta) = \beta$$

and we are done. □

Next, we estimate the virtual entropy of infinite minimal sets $M \subset \mathbb{D}$ for f. When M is disjoint from the kernel $\mathfrak{K}(\mathcal{H})$, the coding map $\mathbf{a}\colon M \to \Omega(M)$ immediately gives the estimate $\operatorname{ent}(\Omega(M)) \leq \operatorname{ent}(M)$. We wish to establish this, for $M \subset \mathbb{D}$ an infinite minimal set, even when M intersects $\mathfrak{K}(\mathcal{H})$, provided that some image of $\mathfrak{K}(\mathcal{H})$ is (at most) countable. Note that since M must be uncountable, it is not contained in $\mathfrak{K}(\mathcal{H})$. Our estimate employs a formula of Bowen together with an easy but nonstandard result about sojourn times inside infinite minimal sets.

Suppose $F\colon Y \to Y$ has $f\colon X \to X$ as a factor via the projection $\pi\colon Y \to X$ ($\pi \circ F = f \circ \pi$). For each point $x \in X$, the fiber $\pi^{-1}[x] \subset Y$ is not in general F-invariant (rather, $F(\pi^{-1}[x]) = \pi^{-1}[f(x)]$), but we can still consider the maximal cardinality $\mathrm{maxsep}[n, \varepsilon, \pi^{-1}[x]]$ of (n, ε)-separated subsets of $\pi^{-1}[x]$, and then define

$$\mathrm{ent}(\pi^{-1}[x]) := GR\{\mathrm{maxsep}[n, \varepsilon, \pi^{-1}[x]]\}.$$

Bowen [Bow71, Theorem 17] showed that, with this definition,

$$\mathrm{ent}(F) \leq \mathrm{ent}(f) + \sup_{x \in X} \mathrm{ent}(\pi^{-1}[x]). \tag{15}$$

Given $M \subset \mathbb{D}$ f-invariant we will take

$$Y := \{(x, \mathbf{a}) \mid x \in M,\ \mathbf{a} \in \Omega(M)\} \subset M \times \Omega_m$$

and

$$F(x, \mathbf{a}) := (f(x), \sigma(\mathbf{a})).$$

Since the shift map $\sigma\colon \Omega(M) \to \Omega(M)$ is a factor of F via $(x, \mathbf{a}) \mapsto \mathbf{a}$, we have

$$\mathrm{ent}(\Omega(M)) \leq \mathrm{ent}(F).$$

But applying (15) to the factor map $(x, \mathbf{a}) \mapsto x$, we also have

$$\mathrm{ent}(F) \leq \mathrm{ent}(M) + \sup_{x \in M} \mathrm{ent}(\Omega(x)).$$

Thus, it will suffice to show that, when $M \subset \mathbb{D}$ is an infinite minimal set, then $\mathrm{ent}(\Omega(x)) = 0$ for every $x \in M$. This will follow from an easy but nonstandard result about sojourn times (recall (8)).

Lemma 4.6. *Suppose M is an infinite minimal set and $K \subset M$ is a closed, countable subset. For $x \in M$, $n \in \mathbb{N}$, define*

$$\kappa_n(x) := \theta(x, n, K).$$

Then

$$\frac{1}{n}\kappa_n(x) \to 0 \text{ as } n \to \infty.$$

Proof. Suppose not, and pick $\varepsilon > 0$ and a sequence $n_i \to \infty$ such that

$$\kappa_{n_i}(x) \geq n_i \varepsilon.$$

For each i, form the measure on M

$$\mu_i := \frac{1}{n_i} \sum_{j=1}^{n_i} \delta_{f^j(x)}$$

where $\delta_{f^j(x)}$ is the point-mass at $f^j(x)$. For any set $A \subset M$,

$$\mu_i(A) = \frac{1}{n_i}\theta(x, n_i, A).$$

By standard arguments [DGS76, prop. 3.8] the weak accumulation points of the sequence of measures μ_i is a nonempty set of f-invariant probability measures on M. But for each i,

$$\mu_i(K) \geq \varepsilon$$

so for any accumulation measure μ, we have $\mu(K) \geq \varepsilon > 0$. Since K is countable, μ must have atoms in K. This is impossible, since no point of M ever returns to itself and $\mu(f(x)) \geq \mu(x)$ for all points x. ∎

Using this, we obtain

Lemma 4.7. *Suppose that for some $k \in \mathbb{N}$, $f^k(\mathfrak{K}(\mathcal{H}))$ is (at most) countable. Then for any infinite minimal set $M \subset S$,*

$$\mathrm{ent}(\Omega(M)) \le \mathrm{ent}(M).$$

Proof. Since $\mathfrak{K}(\mathcal{H})$ is closed, we can take $K = f^k(\mathfrak{K}(\mathcal{H})) \cap M$ in Lemma 4.6 to conclude that $\frac{1}{n}\kappa_n(x) \to 0$. But since a point of M outside $\mathfrak{K}(\mathcal{H})$ has a unique address, we have

$$\mathrm{card}\,\Omega(x)(n) \le m^{\theta(x,n,M\cap\mathfrak{K}(\mathcal{H}))} \le m^{\kappa_{n+k}(x)}$$

and the growth rate of this is

$$\mathrm{ent}(\Omega(x)) \le \lim_{n\to\infty} \frac{\kappa_{n+k}(x)}{n+k} \cdot \frac{n+k}{n} \cdot \log m = 0.$$

□

4.3. **Endgame.** This section is devoted to the proof of the following result.

Theorem 4.8. *Suppose $\mathcal{H}$ is a nontrivial closed m-shift system for $f\colon S \to S$ such that*

(1) *$f^k(\mathfrak{K}(\mathcal{H}))$ is (at most) countable for some $k \in \mathbb{N}$;*
(2) *$\mathfrak{Z}_0(\mathcal{H})$ is (at most) finite, and every periodic orbit in $\mathfrak{Z}_0(\mathcal{H})$ locally divides $\mathcal{H}$.*

Then

$$h_{top}(f) \ge \log m.$$

In §§5 and 6, we will show that this situation holds for any m-fold circle map with no small preimage sets, thus proving Theorem 1.4. Our proof of Theorem 4.8 relies on a contradiction between Remark 3.1 and the virtual entropy estimates of §§4.1-4.2. To this end, we first establish some "separation" results for n-itinerary sets.

Remark 4.9. *Suppose $x \in S$ and $A \subset \Omega_m$ such that for some $j, k \in \mathbb{N}$*

$$\Omega(f^j(x))(k) \cap \sigma^j[A](k) = \emptyset.$$

Then

$$\Omega(x)(k+j) \cap A(k+j) = \emptyset.$$

Proof. For any $w = w_0 \dots w_{k+j-1} \in \Omega(x)(k+j)$ and $\mathbf{a} = a_0a_1... \in A$, $\sigma^j(w) = w_j...w_{k+j-1} \in \Omega(f^j(x))(k)$ and $\sigma^j(\mathbf{a}) = a_j...a_{k+j-1} \in \sigma^j[A](k)$, hence $w_i \neq a_i$ for at least one i in the range $j \le i < k+j$; thus $w \neq \mathbf{a}(k+j)$. □

Lemma 4.10. *Suppose $A \subset \Omega_m$ is closed and shift-invariant, and $y \in \omega(x) \subset S$ with $\Omega(y) \cap A = \emptyset$. Then there exists a neighborhood U of x and $k \in \mathbb{N}$ such that for all $x' \in U$*

$$\Omega(x')(k) \cap A(k) = \emptyset.$$

Proof. Since $\Omega(y)$ and A are disjoint closed subsets of Ω_m, there exists $k_1 \in \mathbb{N}$ such that $\Omega(y)(k_1) \cap A(k_1) = \emptyset$, or $y \notin \Pi(A(k_1))$. Let U_1 be a neighborhood of y disjoint from $\Pi(A(k_1))$, and pick j with $f^j(x) \in U_1$. Then $U := f^{-j}[U_1]$ is a neighborhood of x, and (since $\sigma^j[A] \subset A$) for every $x' \in U$, $\Omega(f^j(x'))(k_1) \cap \sigma^j[A](k_1) = \emptyset$, so by Remark 4.9

$$\Omega(x')(k_1+j) \cap A(k_1+j) = \emptyset.$$

Thus $k = k_1 + j$ works. □

Using these results, we establish the following, which will be the central point in our proof of Theorem 4.8.

Proposition 4.11. *Suppose $\mathcal{H}$ satisfies the hypotheses of Theorem 4.8, and that $\Gamma \subset \Omega_m$ is a shift-minimal set with the property that*

$$\mathrm{ent}(\Gamma) > \max\{\log(m-1), \mathrm{ent}(\Omega(M))\}$$

for every f-minimal set M disjoint from $\mathfrak{Z}_0(\mathcal{H})$. Then there exists $k \in \mathbb{N}$ such that every point $x \in S$ for which $\Omega(x)(k) \cap \Gamma(k) \neq \emptyset$ must satisfy $f^{k-1}(x) \in \mathfrak{Z}_0(\mathcal{H})$ a periodic point.

Proof. We construct for every $x \in S$ a neighborhood $U(x)$, and an associated integer $k(x)$, such that every $x' \in U(x)$ for which $f^{k(x)-1}(x')$ is not a periodic point in $\mathfrak{Z}_0(\mathcal{H})$ satisfies $\Omega(x')(k(x)) \cap \Gamma(k(x)) = \emptyset$. We treat three cases; even though the second and third need not be mutually exclusive, this presents no problem.

If $x \notin \mathbb{D}$, pick $k(x)$ so that $f^{k(x)}(x) \notin \mathbb{H}$, and a neighborhood $U(x)$ of x for which $f^{k(x)}(U(x)) \cap \mathbb{H} = \emptyset$, and hence $\Omega(x')(k(x)) = \emptyset$ for all $x' \in U(x)$.

If $\omega(x)$ contains a minimal set M which is not contained in $\mathfrak{Z}_0(\mathcal{H})$, then since $\mathrm{ent}(\Omega(M)) < \mathrm{ent}(\Gamma)$ and Γ is minimal, $\Omega(M)$ is disjoint from Γ. It follows by Lemma 4.10 with y any element of M and $A = \Gamma$, that we can find $U(x)$ and $k(x)$ so that $\Omega(x')(k(x)) \cap \Gamma(k(x)) = \emptyset$ for all $x' \in U(x)$.

If $\omega(x) \cap \mathfrak{Z}_0(\mathcal{H}) \neq \emptyset$, let P be any periodic orbit contained in $\omega(x) \cap \mathfrak{Z}_0(\mathcal{H})$, and pick V, Λ as in Lemma 4.2. Since $\mathrm{ent}(\Lambda) \leq \log(m-1) < \mathrm{ent}(\Gamma)$, Λ is disjoint from Γ, and hence $\Lambda(k_0) \cap \Gamma(k_0) = \emptyset$ for some $k_0 \in \mathbb{N}$. But since $P \subset \omega(x)$, there exists $k_1 \in \mathbb{N}$ so that $f^{k_1+j}(x) \in V$ for $0 \leq j < k_0$, and a neighborhood $U(x)$ so that the same holds true for every $x' \in U(x)$; let $k(x) := k_0 + k_1$. For any $x' \in U(x)$ with $f^{k(x)-1}(x') \notin P$, we have $f^{k_1+j}(x') \in V \setminus \mathfrak{Z}_0(\mathcal{H})$ for $0 \leq j < k_0$, hence $\Omega(f^{k_1}(x'))(k_0) \subset \Lambda(k_0)$. But the latter is disjoint from $\Gamma(k_0)$, and hence $\Omega(x')(k(x))$ is disjoint from $\Gamma(k(x))$ by Lemma 4.10.

Since $\omega(x)$ always contains some minimal set, these cases are exhaustive, and so $\{U(x) \,|\, x \in S\}$ form an open cover of S. Let $\{U(x_i) \,|\, i = 1, \ldots, N\}$ be a finite subcover, and set

$$k := \max_{i=1,\ldots,N} k(x_i).$$

Then we clearly have the desired conclusion with this value of k. □

Proof of Theorem 4.8:

Let β be given by (13). We will show that for $0 < \varepsilon < \log m - \beta$ f has minimal sets M with

$$\mathrm{ent}(M) \geq \log m - \varepsilon. \tag{16}$$

By [Gri73], Ω_m contains shift-minimal sets with entropy arbitrarily near $\log m$, so we can find Γ_ε minimal with

$$\mathrm{ent}(\Gamma_\varepsilon) > \log m - \varepsilon > \beta > \log(m-1).$$

If $M \subset \mathbb{H} \setminus \mathfrak{Z}_0(\mathcal{H})$ has $\mathrm{ent}(M) < \log m - \varepsilon$, then by Proposition 4.5 and Lemma 4.7, $\mathrm{ent}(\Omega(M)) < \log m - \varepsilon$. Thus, if no minimal set M has $\mathrm{ent}(M) \geq \log m - \varepsilon$, then Proposition 4.11 says that for some $k \in \mathbb{N}$, $\Omega(x)(k) \cap \Gamma_\varepsilon(k) = \emptyset$ for every x with $f^{k-1}(x)$ not a periodic point in $\mathfrak{Z}_0(\mathcal{H})$. But Remark 3.1 says that every $w \in \Gamma_\varepsilon(k)$ belongs to some $\Omega(x)(k)$ for a point with $f^{k-1}(x)$ not a periodic point in $\mathfrak{Z}_0(\mathcal{H})$, a contradiction. This establishes the existence of minimal sets satisfying (16). Thus $\text{ent}(f) \geq \log m - \varepsilon$, and since $\varepsilon > 0$ can be chosen arbitrarily small in (16), (2) follows. □

5. The Kernel

In this section, we establish, for $f\colon S \to S$ cocountably m-fold, the existence of m-sections satisfying Theorem 4.8, Condition 1 with $k = 1$.

5.1. **Regular Values.** Suppose f is a map continuously defined on a full neighborhood of $x \in S$ and $y = f(x)$. We say x is a **nonminimal** (*resp.* **nonmaximal**) preimage of y if there exist points x' arbitrarily near x with $f(x') < y$ (*resp.* $f(x') > y$)—in other words, if every full neighborhood of x maps to a left (*resp.* right) neighborhood of y.

Lemma 5.1. *Let $I \subset S$ be a closed interval and f continuous on I with $Y := f(I)$ an interval. Then every $y \in \text{int } Y$ has at least one nonminimal and at least one nonmaximal preimage in* int I.

Proof. The lemma is vacuous if Y is a single point, so replacing I with a subinterval with the same image we can assume that $I = [a, b]$ and the endpoints of Y are $f(a) \neq f(b)$. For $y \in \text{int } Y$, the set $f^{-1}[y] \cap I$ is a closed nonempty subset of int I, and setting x_- (*resp.* x_+) the minimum (*resp.* maximum) element of this set, we can find strictly monotonic sequences $\{x_i^-\}_{i=0}^{\infty}$ (*resp.* $\{x_i^+\}_{i=0}^{\infty}$) in int I increasing (*resp.* decreasing) to x_- (*resp.* to x_+). If $f(a) < f(b)$, then $f(x_i^-) < y$ (*resp.* $f(x_i^+) > y$) for all i, so x_- (*resp.* x_+) is nonminimal (*resp.* nonmaximal). If $f(a) > f(b)$, then x_+ is nonminimal and x_- is nonmaximal. □

Given $m \in \mathbb{N}$, we will call $y \in S$ a **left m-regular** (*resp.* **right m-regular**) value for the continuous map $f\colon S \to S$ if y has at least m nonminimal (*resp.* nonmaximal) preimages. The set of left m-regular (*resp.* right m-regular) values of f will be denoted $\mathfrak{C}_m(f, \ell)$ (*resp.* $\mathfrak{C}_m(f, r)$), and their intersection will be denoted $\mathfrak{C}_m(f)$.

Lemma 5.2. *If y is left (resp. right) m-regular, then there exists a left (resp. right) neighborhood Y of y with*

$$\text{int } Y \subset \mathfrak{C}_m(f).$$

Proof. We prove the left-regular case, leaving the analogous right-regular proof to the reader.

Pick x_j, $j = 1, \ldots, m$ distinct nonminimal preimages of y and for each a full neighborhood U_j so that the sets U_j, $j = 1, \ldots, m$ are pairwise disjoint; we can assume (shrinking U_j if necessary) that $f(U_j)$ is an interval, which by construction contains a left neighborhood $Y_j = J_-(y, \varepsilon_j)$ of y. By Lemma 5.1, each point interior to $f(U_j)$ has at least one nonminimal and one nonmaximal preimage in U_j. Thus, any point y' interior to the left neighborhood of y

$$Y := \bigcap_{j=1}^{m} Y_j = J_-(y, \varepsilon) \quad (\varepsilon = \min \varepsilon_j > 0)$$

has at least one nonminimal and one nonmaximal preimage in *each* U_j, $j = 1, \ldots, m$, and hence belongs to $\mathfrak{C}_m(f)$. □

A variant of this argument gives

Lemma 5.3. *If $f\colon S \to S$ is a nonconstant continuous map and $y \in S$ is a point whose preimage set has at least $m \in \mathbb{N}$ components, then there exists a nontrivial interval Y with y an endpoint such that*

$$\operatorname{int} Y \subset \mathfrak{C}_m(f).$$

Proof. $f^{-1}[y]$ is a closed proper subset of S with at least m components, so its complement $S \setminus f^{-1}[y]$ also has at least m components, each of the form (x_+, x_-), where $x_+, x_- \in f^{-1}[y]$.

Associate to any such component a pair of closed intervals with disjoint interiors of the form $[x_+, x_+ + \delta]$, $[x_- - \delta, , x_-]$, noting that each maps to a (nontrivial) interval with one endpoint at y. A choice of m components of $S \setminus f^{-1}[y]$ yields $2m$ closed intervals K_j with disjoint interiors, such that for each $j = 1, \ldots, 2m$, $f(K_j)$ contains an immediate one-sided neighborhood $Y_j = J_\pm(y, \varepsilon_j)$ of y, and by Lemma 5.1 every interior point of Y_j has at least one nonminimal and one nonmaximal preimage interior to K_j. Renumbering, we can assume that the first m intervals Y_j, $j = 1, \ldots, m$ are all left neighborhoods (*resp.* all right neighborhoods) of y, and as in the previous argument, each y' interior to the left (*resp.* right) neighborhood of y

$$Y := \bigcap_{j=1}^{m} Y_j$$

has one nonminimal and at least one nonmaximal preimage interior to *each* K_j, so

$$\operatorname{int} Y \subset \mathfrak{C}_m(f).$$

□

Using these lemmas, we can show that for a cocountably m-fold map $f\colon S \to S$, the set $\mathfrak{C}_m(f)$ is very large.

Proposition 5.4. *For any cocountably m-fold map $f\colon S \to S$, $\mathfrak{C}_m(f)$ is an open cocountable set.*

Proof. By Lemma 5.2, any point of $\mathfrak{C}_m(f, \ell)$ (*resp.* $\mathfrak{C}_m(f, r)$) is the right (*resp.* left) endpoint of a nonempty open interval contained in $\mathfrak{C}_m(f)$; it follows that $\mathfrak{C}_m(f)$ is open.

Since f is m-fold on a cocountable subset $Y \subset S$, any preimage set $f^{-1}[y]$, $y \in Y$ with fewer than m components has nonempty interior, and since they are pairwise disjoint, there are at most countably many such preimage sets. Thus by reducing Y we have a cocountable set such that for every $y \in Y$, $f^{-1}[y]$ has at least m components.

If $f^{-1}[y]$ has at least m components, by Lemma 5.3 there is a component $Y(y)$ of $\mathfrak{C}_m(f)$ whose closure contains y; if $y \notin \mathfrak{C}_m(f)$ it is an endpoint of $Y(y)$. Since $\mathfrak{C}_m(f)$ is open, it has at most countably many components, and collectively these have at most countably many endpoints.

Thus, $S \setminus \mathfrak{C}_m(f)$ is contained in the union of two at most countable sets, proving the proposition. □

5.2. **Regular Sections.** We define several calibrations of the "spread" of an m-section ψ on $Y \subset S$. Given $y \in Y$, set

$$\delta(\psi, y) := \min_{1\le j<j'\le m} \mathrm{dist}(\psi_j(y), \psi_{j'}(y))$$

and then the *mesh* of ψ on the set $U \subset Y$ is

$$\Delta(\psi, U) := \inf\{\delta(\psi, y) \mid y \in U\}.$$

Also, the distance on the product $\mathsf{X}_{i=1}^m S$ defines a distance between the values of ψ at two points $y, y' \in Y$

$$\mathrm{mdist}(\psi(y), \psi(y')) := \max_{j=1,\dots,m} \mathrm{dist}(\psi_j(y), \psi_j(y'))$$

and the *variation* of ψ across a set $U \subset Y$ is

$$\|\psi\|_U := \sup\{\mathrm{mdist}(\psi(y), \psi(y')) \mid y, y' \in U\}.$$

Manipulating the relation

$$\mathrm{dist}(\psi_j(y), \psi_{j'}(y)) \le \mathrm{dist}(\psi_j(y), \psi_j(y')) + \mathrm{dist}(\psi_j(y'), \psi_{j'}(y')) + \mathrm{dist}(\psi_{j'}(y'), \psi_{j'}(y))$$

we can show the following

Remark 5.5. *For any m-section ψ on Y and any subset $U \subset Y$ we have, if $y \in U$,*

$$0 \le \delta(\psi, y) - \Delta(\psi, U) \le 2\|\psi\|_U.$$

By definition, $f\colon S \to S$ is m-fold precisely if it has at least one m-section ψ on S, but in general ψ cannot be chosen to be continuous. However, an elaboration of Lemma 5.2 allows us, using regularity conditions, to control $\mathrm{mdist}(\psi(y), \psi(y'))$ for all y' close to y on one side. Call an m-section ψ on $Y \subset S$ **left regular** (*resp.* **right regular**) if $\psi_j(y)$ is a nonminimal (*resp.* nonmaximal) preimage of y for every $y \in Y$ and $j = 1, \dots, m$. When $Y = \{y\}$, we shall refer to regularity at y. Of course, $y \in \mathfrak{C}_m(f, \ell)$ (*resp.* $y \in \mathfrak{C}_m(f, r)$) if there is a left (*resp.* right) regular m-section at y, but even if $y \in \mathfrak{C}_m(f)$, there need not be an m-section which is simultaneously both left and right regular.

Lemma 5.6. *Suppose $\delta > 0$ and $\psi(y)$ is a left (resp. right) regular m-section of $f\colon S \to S$ at $y \in S$. Then there exists a right (resp. left) neighborhood $Y(y)$ of y and two m-sections, λ and ρ, on $Y(y)$ such that*

1. *$\lambda(y) = \rho(y) = \psi(y)$;*
2. *λ is left regular and ρ is right regular on $\mathrm{int}\, Y(y)$[3];*
3. *For every $y' \in Y(y)$, $\mathrm{mdist}(\lambda(y), \lambda(y')) < \delta$ and $\mathrm{mdist}(\rho(y), \rho(y')) < \delta$.*

Proof. Assume without loss of generality that

$$0 < \delta < \Delta(\psi, \{y\})$$

and for $j = 1, \dots, m$ let

$$U_j := (\psi_j(y) - \frac{1}{2}\delta, \psi_j(y) + \frac{1}{2}\delta).$$

The sets U_j are nonempty open, pairwise disjoint sets, and as in the proof of Lemma 5.2 there exists $\varepsilon > 0$ such that for every $j = 1, \dots, m$, every point y' interior to $Y(y) = J_-(y, \varepsilon)$ (*resp.* $Y(y) = J_+(y, \varepsilon)$) has at least one nonminimal preimage $\lambda_j(y') \in U_j$ and at least one nonmaximal preimage $\rho_j(y') \in U_j$. This easily leads to the m-sections λ and ρ required by the lemma. □

[3]Note that the regularity of at least one of these m-sections extends to $Y(y)$.

We wish to "globalize" this local lemma; to this end, we need the following elementary topological results (*cf* [Bob04, Lemma 2.2]):

Lemma 5.7. *Suppose $\mathcal{J}$ is a collection of nontrivial intervals in S; set*

$$U := \bigcup_{J\in\mathcal{J}} \text{int } J$$

and

$$\tilde{U} := \bigcup_{J\in\mathcal{J}} J$$

Then[4]

1. *U is a cocountable (open) subset of $\tilde{U}$;*
2. *there exists a cocountable open subset V of U such that each component of V is contained in some element of $\mathcal{J}$.*

Proof. (1) Given $x \in \tilde{U} \setminus U$, pick $J \in \mathcal{J}$ containing x; then x is an endpoint of J, and of the component of U containing int J. Since U has at most countably many components, this proves the first assertion.

(2) Second-countability allows us to restrict attention to an at most countable subcollection $J_1, J_2, \ldots \in \mathcal{J}$ whose interiors cover U, and then define

$$V_i = \text{int } J_i \setminus \bigcup_{i'<i} \text{clos } J_{i'}.$$

Clearly, V_i is a cofinite open subset of $J_i \cap U$, and the sets V_i are pairwise disjoint; it follows easily that

$$V = \bigcup_i V_i$$

is as required.

□

Proposition 5.8. *Suppose $f\colon S \to S$ is a cocountably m-fold map. Then there is a cocountable open subset $Y \subset \mathfrak{C}_m(f)$ and a left-regular (resp. right-regular) m-section ψ of f on Y such that, for every component Y_i of Y*

$$\Delta(\psi, Y_i) > 0 \tag{17}$$

$$\|\psi\|_{Y_i} < \frac{1}{2}\Delta(\psi, Y_i). \tag{18}$$

Proof. We prove the left-regular version.

For each $y \in \mathfrak{C}_m(f)$, use Lemma 5.6 to pick a left-regular m-section λ_y on a left neighborhood $J(y)$ of y such that

$$\|\lambda_y\|_{J(y)} < \frac{1}{4}\delta(\lambda_y, \{y\}).$$

Then, using Remark 5.5, we have

$$\Delta(\lambda_y, J(y)) \geq \delta(\lambda_y, \{y\}) - 2\|\lambda_y\|_{J(y)} > \frac{1}{2}\delta(\lambda_y, \{y\}).$$

Note that this is strictly positive, and also is bounded below by $2\|\lambda_y\|_{J(y)}$.

[4]Note that U need not equal int $\tilde{U}$.

Now apply Lemma 5.7 to

$$\mathcal{J} := \{J(y) \mid y \in \mathfrak{C}_m(f)\}$$

to obtain a cocountable open subset Y of $\mathfrak{C}_m(f)$ such that each component Y_i of Y is contained in some element $J(y_i) \in \mathcal{J}$. (Note: we do *not* claim that $y_i \in Y_i$.) Pick such an interval $J(y_i)$ and let $\psi = \lambda_{y_i}$ on Y_i. Then we have

$$\Delta(\psi, Y_i) \geq \Delta(\lambda_{y_i}, J(y_i)) > 2\|\lambda_{y_i}\|_{J(y_i)} \geq 2\|\psi\|_{Y_i},$$

(where the second inequality follows from our observation above). Since the middle term is positive, we obtain the first assertion, and the outside inequalities give the second. □

These results can now be applied to give us information about the kernel of the shift system associated to an appropriate m-section.

Proposition 5.9. *Suppose that, as in Proposition 5.8, ψ is an m-section of f on the cocountable open set $Y \subset S$ such that every component Y_i of Y satisfies*

$$\|\psi\|_{Y_i} < \frac{1}{2}\Delta(\psi, Y_i).$$

Let $\mathcal{H} = \{H_i \mid i = 1, \ldots, m\}$ be the closed m-shift system associated to ψ.

Then $f(\mathfrak{K}(\mathcal{H})) \cap Y = \emptyset$, so $f(\mathfrak{K}(\mathcal{H}))$ is at most countable.

Proof. Suppose $x \in H_{j_1} \cap H_{j_2}$, $j_1 \neq j_2$, and pick sequences $y_i, y_i' \in Y$ such that

$$x = \lim \psi_{j_1}(y_i) = \lim \psi_{j_2}(y_i').$$

By continuity of f,

$$f(x) = \lim f(\psi_{j_1}(y_i)) = \lim y_i$$

and similarly

$$f(x) = \lim f(\psi_{j_2}(y_i')) = \lim y_i'.$$

Suppose $f(x) \in Y$; let U be the component of Y containing $f(x)$, and pick i so that

$$\operatorname{dist}(\psi_{j_1}(y_i), x) < \frac{1}{4}\Delta(\psi, U) \text{ and } \operatorname{dist}(\psi_{j_2}(y_i'), x) < \frac{1}{4}\Delta(\psi, U).$$

Then

$$\begin{aligned}
\Delta(\psi, U) &\leq \operatorname{dist}(\psi_{j_1}(y_i), \psi_{j_2}(y_i)) \\
&\leq \operatorname{dist}(\psi_{j_1}(y_i), \psi_{j_2}(y_i')) + \operatorname{dist}(\psi_{j_2}(y_i'), \psi_{j_2}(y_i)) \\
&\leq \operatorname{dist}(\psi_{j_1}(y_i), x) + \operatorname{dist}(x, \psi_{j_2}(y_i')) + \operatorname{dist}(\psi_{j_2}(y_i'), \psi_{j_2}(y_i)) \\
&\leq \|\psi\|_U + \tfrac{1}{2}\Delta(\psi, U) \\
&< \Delta(\psi, U),
\end{aligned}$$

a contradiction. □

6. The Center

In this section, we show how to insure Condition 2 of Theorem 4.8 for any cocountably m-fold map with no small preimage sets. We begin by transferring our study from the circle to the interval.

Suppose $f\colon S \to S$ is a continuous map with no small preimage sets. Recall that f has a lift to the universal cover, a continuous map $F\colon \mathbb{R} \to \mathbb{R}$ satisfying $\pi \circ F = f \circ \pi$, where $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the quotient map. F satisfies $F(x + n) = F(x) + nd$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, where $d := F(1) - F(0)$ is the degree of f. Define $G\colon I := [0,1] \to [0,1)$ by

$$G(x) = F(x) \pmod{\mathbb{Z}}.$$

G is continuous at $x \in I$ if $F(x)$ is not an integer (*i.e.*, if $G(x) \neq 0$). Let

$$D := [0,1] \setminus G^{-1}[0]$$

a (relatively) open subset of I. For $u \in D$, denote by $D(u)$ the component of D containing u, and

$$g = g_u\colon \text{clos } D(u) \to I$$

the (unique) continuous extension of $G|D(u)$ to clos $D(u)$. We call g_u the **branch** of G at $u \in D$ and $D(u)$ its **branch component**. If 0 (*resp.* 1) is an endpoint of some branch component, we label that component D_0 (*resp.* D_1). Note that if an endpoint of I maps to the interior of I then $G(0) = G(1) \in \text{int } I$ and D_0 and D_1 both exist.

Remark 6.1. (1) *The maps g_u are uniformly branchwise continuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x' \in D(x)$ and $|x - x'| < \delta$ implies $|g(x) - g(x')| < \varepsilon$.*
(2) *Since f is surjective, $G^{-1}[0]$ is nonempty, and the absence of small preimage sets insures that there are at least two distinct branch components.*
(3) *Any endpoint e of $D(u)$ (with the possible exception of $e \in \partial I := \{0,1\}$) has $g_u(e) \in \partial I$.*
(4) *In particular, if $x_1 < x_2$ do not lie in a common component of D, then we can find points x_1', x_2' with $x_1 \leq x_1' \leq x_2' \leq x_2$ such that*
- *if $x_i \in D$ then x_i' is an endpoint of $D(x_i)$;*
- *hence $g([x_1, x_1'])$ (resp. $g([x_2', x_2])$) contains an interval of the form $[y_1, e_1]$ (resp. $[e_2, y_2]$) where $y_i = G(x_i) \pmod{\mathbb{Z}}$ and $e_i \in \partial I$.*

For any $y \in \text{int } I$, set

$$m_y := \min G^{-1}[y]; \quad M_y := \max G^{-1}[y].$$

Noting that for $x, x' \in I$,

$$\text{dist}(\pi(x), \pi(x')) = \min\{|x - x'|, 1 - |x - x'|\} \leq \frac{1}{2},$$

we see that if f has no small preimage sets, then

$$M_y - m_y \geq \alpha \text{ for all } y \in \text{int } I. \tag{19}$$

Let

$$\begin{aligned} S_- &:= \text{clos } \{m_y \mid y \in Y\} \subset I \\ S_+ &:= \text{clos } \{M_y \mid y \in Y\} \subset I, \\ S_c &:= S_- \cap S_+. \end{aligned}$$

The following observations are immediate consequences of (19) and the definitions.

Remark 6.2. (1) *$S_- \subseteq [0, 1-\alpha]$, $S_+ \subseteq [\alpha, 1]$, so $S_c \subseteq [\alpha, 1-\alpha]$.*
(2) *If $m_{y_i} \to u \in D$ (resp. $M_{y_i} \to u$), then $y_i \to y := G(u) \in \text{int } I$.*
(3) *If $y_i \to y \in \text{int } I$, then any accumulation point of $\{m_{y_i}\}$ (resp. of $\{M_{y_i}\}$) is $\leq M_y - \alpha$ (resp. $\geq m_y + \alpha$).*
(4) *If $u \in S_c \cap D$, then setting $y := G(u)$,*

(a) $m_y + \alpha \le u \le M_y - \alpha$, *so* $u \in [\alpha, 1-\alpha]$
(b) *hence, there exist sequences* y_i, $\bar{y}_i$, *distinct from* y, *with* $u = \lim m_{y_i} = \lim M_{\bar{y}_i}$.

Lemma 6.3. *Suppose* $\tilde{y}$ *is between* y' *and* y'', *all belonging to* int I.

(1) *If* $m_{\tilde{y}} < m_{y'} < m_{y''}$, *then* $D(m_{y'}) = D(m_{\tilde{y}}) = D_0$, *and* $G(0)$ *is strictly between* y' *and* y''.
(2) *If* $M_{\tilde{y}}$ *is between* $m_{y'}$ *and* $m_{y''}$, *and* $|m_{y'} - m_{y''}| < \alpha$, *then* $G(0)$ *is strictly between* y' *and* y''.
(3) *If* $M_{\tilde{y}} > M_{y'} > M_{y''}$, *then* $D(M_{y'}) = D(M_{\tilde{y}}) = D_1$, *and* $G(1)$ *is strictly between* y' *and* y''.
(4) *If* $m_{\tilde{y}}$ *is between* $M_{y'}$ *and* $M_{y''}$, *and* $|M_{y'} - M_{y''}| < \alpha$, *then* $G(1)$ *is strictly between* y' *and* y''.

Proof. We prove (1) and (2); the proofs of (3) and (4) are analogous.

Proof of (1):

If $D(m_{y'}) \ne D(m_{\tilde{y}})$, set $x_1 = m_{\tilde{y}}$, $x_2 = m_{y'}$ in Remark 6.1(4). Then $x_1' < m_{y'}$ with $g([x_1, x_1']) \cap \{y', y''\} \ne \emptyset$, implying that one of $m_{y'}$, $m_{y''}$ is $\le x_1'$, a contradiction.

Similarly, if $D(m_{\tilde{y}}) \ne D_0$, set $x_1 = 0$, $x_2 = m_{\tilde{y}}$ in Remark 6.1(4). Then $x_2' < x_2 < m_{y'}$ with $g([x_2', x_2]) \cap \{y', y''\} \ne \emptyset$; hence one of $m_{y'}$, $m_{y''}$ is $\le x_2 = m_{\tilde{y}}$, another contradiction.

Finally, if $G(0)$ is not strictly between y' and y'', then $g([0, m_{\tilde{y}}]) \cap \{y', y''\} \ne \emptyset$, a third contradiction. ◊

Proof of (2):

$m_{\tilde{y}} \le M_{\tilde{y}} - \alpha < \min\{m_{y'}, m_{y''}\}$, and (1) applies. ◊

□

These observations allow us to paint a fairly detailed picture of the sequences specified in Remark 6.2(4b).

Proposition 6.4. *Suppose* $u = \lim m_{y_i} = \lim M_{\bar{y}_i}$ *with* y_i *and* $\bar{y}_i$ *all distinct. Set* $y = \lim y_i$ *and* $\bar{y} = \lim \bar{y}_i$. *Then*

(1) *Either* $y = \bar{y} \in$ int I *eventually separates* y_i *from* $\bar{y}_i$, *or* $\{y_i, \bar{y}_i\} = \{0, 1\}$.
(2) *Eventually, all* m_{y_i} *are separated from all* $M_{\bar{y}_i}$ *by* u.

Proof. Proof of (1):

Suppose $y = \bar{y}$ and infinitely many y_i's and $\bar{y}_i$'s lie on the same side of y. Passing to subsequences, we can assume the y_i's and $\bar{y}_i$'s are *interlaced*, say $\bar{y}_i$ is between y_i and y_{i+1}, and y_{i+1} is between $\bar{y}_i$ and $\bar{y}_{i+1}$ for all i. For i large, $|m_{y_i} - u|$ and $|M_{\bar{y}_i} - u|$ are both less than $\frac{\alpha}{2}$. For any such i, Lemma 6.3(1) with $\tilde{y} = \bar{y}_i$, $y' = y_i$ and $y'' = y_{i+1}$ implies that $G(0)$ lies strictly between y_i and y_{i+1}, a condition which cannot hold for more than one value of the index i. Thus if $y = \bar{y}$ then $y \in$ int I eventually separates y_i from $\bar{y}_i$. If $G(u) \in$ int I (*i.e.*, $u \in D$) then by Remark 6.2(2), $y = \bar{y}$, so otherwise y and $\bar{y}$ are both endpoints of I, and neither can separate y_i from $\bar{y}_i$, showing that in this case y and $\bar{y}$ are distinct endpoints of I. ◊

Proof of (2):

Suppose infinitely many m_{y_i}'s and $M_{\bar{y}_i}$'s lie on the same side of u, and pass to interlacing subsequences, taking i large enough that all are within $\frac{\alpha}{2}$ of u. Passing to further subsequences, we can assume that the sequences $\{y_i\}$ and $\{\bar{y}_i\}$ converge monotonically. Consider two cases.

Case 1: : If $y = \bar{y} \in \text{int } I$, so $u \in D$, pick $m_{y_1} < M_{\bar{y}_3} < m_{y_2}$, all in $D(u) \cap (u - \frac{\alpha}{2}, u + \frac{\alpha}{2})$, with y_2 between y_1 and y (so also between y_1 and $\bar{y}_3$, by (1)). Since y_2 is between $y_1 = g(m_{y_1})$ and $\bar{y}_3 = g(M_{\bar{y}_3})$, there exists u_3 between m_{y_1} and $M_{\bar{y}_3}$ with $g(u_3) = y_2$, contradicting $M_{\bar{y}_3} < m_{y_2}$.

Case 2:: If y and $\bar{y}$ are distinct endpoints of I, assume the interlaced sequences $\{m_{y_i}\}$, $\{M_{\bar{y}_i}\}$ are indexed so $M_{\bar{y}_i}$ is between m_{y_i} and $m_{y_{i+1}}$, and $m_{y_{i+1}}$ is between $M_{\bar{y}_i}$ and$M_{\bar{y}_{i+1}}$. Since the sequences $\{y_i\}$, $\{\bar{y}_i\}$ converge monotonically to distinct endpoints of I, we can conclude (for i large) that both y_i and $\bar{y}_i$ lie between y_{i+1} and $\bar{y}_{i+1}$. If $m_{y_{i+1}}$ and $M_{\bar{y}_{i+1}}$ lie in the same branch component, then the interval between them contains preimages of both y_i and $\bar{y}_i$, which contradicts one of the statements that m_{y_i} and $M_{\bar{y}_i}$ lie outside this interval. In particular, successive m_{y_i}'s (*resp.* successive $M_{\bar{y}_i}$'s) lie in different components of D. If the m_{y_i}'s and $M_{\bar{y}_i}$'s increase (*resp.* decrease) to u, then the branch at m_{y_i} (*resp.* $M_{\bar{y}_i}$) cannot hit y_{i+1} (*resp.* $\bar{y}_{i+1}$), so by Remark 6.1(4) its endpoints must map to $\bar{y}$ (*resp.* y). But then a point to the left of $m_{y_{i+1}}$ (*resp.* to the right of $M_{\bar{y}_{i+1}}$) maps to y_{i+1} (*resp.* $\bar{y}_{i+1}$), a contradiction.

$\Diamond$

$\square$

From this we obtain important information about S_c.

Proposition 6.5. (1) *For each $u \in \text{int } I$, there exists $\delta > 0$ such that S_- and S_+ cannot both intersect $(u - \delta, u)$ or both intersect $(u, u + \delta)$.*
(2) *S_c is finite.*
(3) *G is injective on S_c.*
(4) *For each $u \in S_c$, there exists $\delta > 0$ such that $g((u - \delta, u))$ and $g((u, u + \delta))$ intersect, if at all, only at $g(u)$ (and in this case $u \in D$).*

Proof. (1) is immediate from Proposition 6.4(2), and since S_c is compact, (2) follows.

Proof of (3):

Suppose $u < u'$ with $u, u' \in S_c$ and $G(u) = G(u') = y$. If $y \neq 0$ (*i.e.,* $y \in \text{int } I$) then by Proposition 6.4(1) we have sequences $m_{y_i}, M_{\bar{y}_i} \to u$ and $m_{y'_i}, M_{\bar{y}'_i} \to u'$ with y_i and $\bar{y}_i$ (*resp.* y'_i and $\bar{y}'_i$) converging to y from opposite sides.

Case 1:: If y'_i are on the same side of y as y_i, pass to interlaced subsequences. Note that $u < u'$ implies for large i, $m_{y_i} < \min\{m_{y'_i}, m_{y'_{i+1}}\}$, so if y_i is between y'_i and y'_{i+1} then by Lemma 6.3(1), $G(0)$ is strictly between y'_i and y'_{i+1}, which cannot occur for more than one index i.

Case 2:: If y'_i are on the same side of y as $\bar{y}_i$, again passing to interlaced subsequences, y'_i is between $\bar{y}_i$ and $\bar{y}_{i+1}$, but for i large $u' > u$ implies $M_{y'_i} > m_{y'_i} > \max\{M_{\bar{y}_i}, M_{\bar{y}_{i+1}}\}$, so Lemma 6.3(3) forces $G(1)$ to lie between $\bar{y}_i$ and $\bar{y}_{i+1}$, which again cannot occur for more than one index i.

If $y = 0$, a similar argument replacing opposite sides of y with disjoint (relative) neighborhoods of 0 and 1 yields a similar contradiction. $\Diamond$

Proof of (4):

By Proposition 6.4 for $u \in D$ we have m_{y_i} and $M_{\bar{y}_i}$ near u with $y = G(u)$ between y_i and $\bar{y}_i$, so it suffices to show that for $\delta > 0$ sufficiently small, we cannot have $x_1 < x_2$ on the same side of u, with $|x_i - u| < \delta$ and y between $g(x_1)$ and $g(x_2)$. Since for every y' between these values there exists x' between x_1 and x_2 with $g(x') = y'$, if $x_1 < x_2 < u$ (*resp.* $u < x_1 < x_2$) then $m_{y'} \leq x_2$ (*resp.* $M_{y'} \geq x_1$) so u cannot be a limit of m_{y_i}'s (*resp.* of $M_{\bar{y}_i}$'s).

For $G(u) = 0$, we have from Proposition 6.4(1) that $g((u-\delta, u])$ and $g([u, u+\delta))$ are small (relative) neighborhoods of $y \neq \bar{y} \in \partial I$, and hence disjoint. $\Diamond$

$\square$

We wish to transfer these results back to the circle.

Suppose ψ is an m-section of $f\colon S \to S$ on the dense set $Y \subset S \setminus \{\pi(0), f(\pi(0))\}$. Then ψ is **monotone** if

$$\psi_1(y) < \psi_2(y) < \cdots < \psi_m(y) \text{ for all } y \in Y$$

(where this is understood in terms of counterclockwise order on the arc $S \setminus \{\pi(0)\}$), and **spanning** if in addition for each $y \in Y$

$$f^{-1}[y] \subset [\psi_1(y), \psi_m(y)]$$

(which corresponds to $\psi_1(y) = m_y$, $\psi_m(y) = M_y$).

With this terminology, Proposition 6.5 translates to the following result on the structure of the center for monotone, spanning m-sections:

Proposition 6.6. *Suppose $f\colon S \to S$ has no small preimage sets, and ψ is a monotone, spanning m-section for f on a dense set $Y \subset S \setminus \{\pi(0), f(\pi(0))\}$. Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the closed m-shift system defined by*

$$H_i := \text{clos}\ \{\psi_i(y) \,|\, y \in Y\} \quad i = 1, \ldots, m.$$

Then

(1) $\mathfrak{Z}(\mathcal{H}) := \bigcap_{i=1}^{m} H_i$ *is finite;*
(2) $\mathfrak{Z}_0(\mathcal{H}) := \bigcap_{k=0}^{\infty} f^{-k}[\mathfrak{Z}(\mathcal{H})]$ *consists of finitely many periodic orbits which locally divide $\mathcal{H}$: there exists $\delta > 0$ such that for any point $u \in \mathfrak{Z}_0(\mathcal{H})$*
  (a) *u lies between any point of $H_1 \cap (u - \delta, u + \delta)$ and any point of $H_m \cap (u - \delta, u + \delta)$;*
  (b) *if $f^k(u) = u$ then $f^k(J_-(u, \delta)) \cap f^k(J_+(u, \delta)) = \{u\}$.*

Proof. Since π is a homeomorphism between int I and $S \setminus \{\pi(0)\}$, clearly

$$\mathfrak{Z}(\mathcal{H}) \subseteq H_1 \cap H_m \subseteq \pi(S_c) \cup \{\pi(0)\}.$$

Thus, $\mathfrak{Z}(\mathcal{H})$ is finite by Proposition 6.5(2), and since f is one-to-one on $\mathfrak{Z}(\mathcal{H})$ by Proposition 6.5(3), every orbit contained in $\mathfrak{Z}(\mathcal{H})$ is periodic. (2a) and (2b) follow, respectively, from Proposition 6.5, parts (1) and (4). $\square$

We can combine Propositions 5.9 and 6.6 to establish the existence of a section satisfying the conditions required by Theorem 4.8, thus proving Theorem 1.4.

Theorem 6.7. *Suppose $f\colon S \to S$ is cocountably m-fold. Then there exists a cocountable open set $Y \subset S$ and an m-section on Y such that the associated m-shift system $\mathcal{H}$ satisfies*

(1) $f(\mathfrak{K}(\mathcal{H})) \subset S \setminus Y$.
(2) $\mathfrak{Z}(\mathcal{H})$ *is finite.*

(3) $\mathfrak{Z}_0(\mathcal{H})$ *consists of periodic orbits which locally divide* $\mathcal{H}$.

Proof. Using Proposition 5.8, let $Y \subset S$ be a cocountable open set and ψ an m-section on Y satisfying (17) and (18) on each component of Y. Without loss of generality, assume $\pi(0), f(\pi(0)) \in S \setminus Y$ and that ψ is monotone. After cyclic permutation of indices (which does not affect monotonicity) we can assume that for each $y \in Y$,

$$\pi(0) \in (\psi_m(y), \psi_1(y)).$$

By abuse of notation, for $y \in Y$ let m_y (*resp.* M_y) be the point nearest to $\pi(0)$ in $f^{-1}[y] \cap (\pi(0), \psi_1(y)]$ (*resp.* $f^{-1}[y] \cap [\psi_m(y), \pi(0))$) and define a new section $\tilde{\psi}$ on Y by

$$\tilde{\psi}_j(y) := \begin{cases} \psi_j(y) & \text{if } 1 < j < m \\ m_y & \text{if } j = 1 \\ M_y & \text{if } j = m. \end{cases}$$

We claim that the m-shift system $\mathcal{H}$ associated to $\tilde{\psi}$ satisfies our requirements.

Clearly, $\tilde{\psi}$ is spanning, so (2) and (3) follow from Proposition 6.6; in fact, we know that $H_1 \cap H_m$ is a finite set, so without loss of generality we can assume that it and its f-image are disjoint from Y.

Condition (1) is equivalent to the statement that

$$f(H_{j_1} \cap H_{j_2}) \subset S \setminus Y$$

for $1 \leq j_1 < j_2 \leq m$. This holds by assumption for the extreme indices $1 = j_1, j_2 = m$ and by Proposition 5.9 for the strictly intermediate indices $1 < j_1 < j_2 < m$. We are left with the cases where precisely one of the indices is extreme: $1 = j_1 < j_2 < m$ and $1 < j_1 < j_2 = m$. We prove the first case; the other is completely analogous.

Suppose $x \in H_1 \cap H_{j_2}$, with $1 < j_2 < m$ and $f(x) \in Y$. Then there exist $y_i, y_i' \in Y$ with

$$x = \lim \tilde{\psi}_1(y_i) = m_{y_i} = \lim \tilde{\psi}_{j_2}(y_i')$$

and so

$$y := f(x) = \lim y_i = \lim y_i'.$$

Since $f(x) \in Y$, we eventually have all of y_i, y_i' and y in a single component of Y. In any such component, we have for all y_i and y_i'

$$\tilde{\psi}_1(y_i) := m_{y_i} \leq \psi_1(y_i) < \psi_{j_2}(y_i') := \tilde{\psi}_{j_2}(y_i')$$

and hence also

$$x = \lim \psi_1(y_i) = \lim \psi_{j_2}(y_i').$$

By Proposition 5.9 this forces the contradictory property $f(x) \in S \setminus Y$, and we are done. □

In light of Theorem 4.8, Theorem 1.4 is an immediate corollary of Theorem 6.7.

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